ELEMENTS

OF THE

MATHEMATICAL THEOR

OF

ELECTRICITY AND MAGNE

CAMBRIDGE UNIVERSITY PR

C. F. CLAY, MANAGER

LONDON : FETTER LANE, E.C.

NEW YORK : THE MACMILLAN

BOMBAY
CALCUTTA } MACMILLAN AND CO.,
MADRAS

TORONTO : THE MACMILLAN CO.
CANADA, LTD.

TOKYO : MARUZEN-KABUSHIKI-KAIS

ELEMENTS

OF THE

MATHEMATICAL THEORY

OF

ELECTRICITY AND MAGNETISM

BY

SIR J. J. THOMSON, O.M.

M.A., Sc.D., D.Sc., LL.D., PH.D., F.R.S.

MASTER OF TRINITY COLLEGE, CAMBRIDGE
PROFESSOR OF PHYSICS IN THE UNIVERSITY OF CAMBRIDGE
PROFESSOR OF NATURAL PHILOSOPHY IN
THE ROYAL INSTITUTION, LONDON

FIFTH EDITION

First Edition 1895
Second Edition 1897
Third Edition 1904
Fourth Edition 1909
Fifth Edition 1921

PREFACE TO FIRST EDITION

ı the following work I have endeavoured to give an account of t
ndamental principles of the Mathematical theory of Electrici
ıd Magnetism and their more important applications, using or
mple mathematics. With the exception of a few paragraphs
ore advanced mathematical knowledge is required from the reac
ıan an acquaintance with the Elementary principles of t
ifferential Calculus.

It is not at all necessary to make use of advanced analysis
tablish the existence of some of the most important elect
agnetic phenomena. There are always some cases which will yi
very simple mathematical treatment and yet which establi
ıd illustrate the physical phenomena as well as the solution
e most elaborate analysis of the most general cases which cou
given.

The study of these simple cases would, I think, often be
lvantage even to students whose mathematical attainments a
fficient to enable them to follow the solution of the more gene
ses. For in these simple cases the absence of analytical difficult
lows attention to be more easily concentrated on the physi
pects of the question, and thus gives the student a more viv
ea and a more manageable grasp of the subject than he wou
likely to attain if he merely regarded electrical phenome
rough a cloud of analytical symbols.

I have received many valuable suggestions and much help
e preparation of this book from my friends Mr H. F. Newall
rinity College and Mr G. F. C. Searle of Peterhouse who have be

PREFACE TO THE SECOND EI

In this Edition I have through the kindnes
spondents been able to correct a considerable nu
I have also made a few verbal alterations in t
the argument clearer in places where experien
students found unusual difficulties.

J. .

Cavendish Laboratory,
Cambridge.
November, 1897.

PREFACE TO THE THIRD ED

The most important of the alterations made i
new chapter on the properties of moving electri
of these properties may be proved in a simple wa
tant part played by moving charges in Modern
warrant a discussion of their properties in eve
Treatise.

I have much pleasure in thanking Mr G. F.
house for many valuable suggestions, and for his l
the proof sheets of the first five chapters; to M
Trinity College I am indebted for similar as
subsequent chapters.

J. J

Cavendish Laboratory,
Cambridge.
October 4, 1904.

PREFACE TO THE FOURTH EDITION

In this Edition a few additions and corrections have been ma

J. J. THOMSON.

Cavendish Laboratory,
Cambridge.
April 26, 1909.

PREFACE TO THE FIFTH EDITION

In this Edition some additions have been made to the text
some misprints corrected. I wish to thank my son, Mr G. P. Th
son, Fellow and Lecturer of Corpus Christi College, Cambr
for the assistance he has given me in its preparation.

J. J. THOMSON.

Trinity Lodge,
Cambridge.
December 12, 1920.

TABLE OF CONTENTS

ELEMENTS OF THE MATHEMATICA
THEORY OF
ELECTRICITY AND MAGNETISM

CHAPTER I

GENERAL PRINCIPLES OF ELECTROSTATICS

1. Example of Electric Phenomena. Electrificati ectric Field. A stick of sealing-wax after being rubbed v vell dried piece of flannel attracts light bodies such as small pi paper or pith balls covered with gold leaf. If such a ball be nded by a silk thread, it will be attracted towards the sealing-w d, if the silk thread is long enough, the ball will move towa e wax until it strikes against it. When it has done this, howe immediately flies away from the wax; and the pith ball is pelled from the wax instead of being attracted towards it as it fore the two had been in contact. The piece of flannel used to e sealing-wax also exhibits similar attractions for the pith ba d these attractions are also changed into repulsions after the b ve been in contact with the flannel.

The effects we have described are called 'electric' phenomena le which as we shall see includes an enormous number of eff the most varied kinds. The example we have selected, wh ctrical effects are produced by rubbing two dissimilar bo ainst each other, is the oldest electrical experiment known ence.

has been rubbed with a piece of flannel, th
repelled, not merely from the sealing-wax bu
To observe this most conveniently remove t
distance from the sealing-wax and the flanne
these are inappreciable. Now take another pa
D, and let them strike against the flannel; C
be repelled from each other when they are pla
take the ball A and place it near C; A an
attracted towards each other. Thus, a ball
sealing-wax is repelled from another ball w
treated, but is attracted towards a ball wh
with the flannel. The electricity on the balls
different from that on the balls C and D, fo
pelled from B it is attracted towards D, whi
towards B and repelled from D; thus when th
ball C is repelled and *vice versâ*.

The state of the ball which has touched
one of *positive electrification*, or the ball is sa
fied; the state of the ball which has touche
to be one of *negative electrification*, or the ba
electrified.

We may for the present regard 'positive
ventional terms, which when applied to el
nothing more than the two states of electri
As we proceed in the subject, however, we
of these terms is justified, since the propertie
electrification are, over a wide range of ph
the properties of the signs *plus* and *minus*

The two balls A and B must be in simila
since they have been similarly treated; the t

is either repelled from the electrified body or attract
In the former case the electrification is negative, in the l

A method, which is sometimes convenient, of dete
the electrification of a body is positive or negative is t
a mixture of powdered red lead and yellow sulphur w
well shaken; the friction of the one powder against the
fies both powders, the sulphur becoming negatively,
positively electrified. If now we dust a negatively ele
with this mixture, the positively electrified red lead w
surface, while the negatively electrified sulphur will
tached, so that if we blow on the powdered surface th
come off while the red lead will remain, and thus the s
coloured red: if a positively electrified surface is treat
it will become yellow in consequence of the sulphur st

3. Electrification by Induction. If the negativ
stick of sealing-wax used in the preceding experiment
to, but not touching, one end of an elongated piece of m
entirely on glass or ebonite stems, and if the metal is du
the mixture of red lead and sulphur, it will be found,
off the loose powder, that the end of the metal nearest
wax is covered with the yellow sulphur, while the end
is covered with red lead, showing that the end of the
the negatively electrified stick of sealing wax is posit
remote from it negatively, electrified. In this experim
which has neither been rubbed nor been in contact with
body, is said to be electrified by *induction*; the electrif
metal is said to be *induced* by the electrification on the st
wax. The electrification on the part of the metal ne
is of the kind opposite to that on the wax, while the

or a more elaborate instrument, called an ele
generally used.

A simple form of electroscope, called the g
represented in Fig. 1. It consists of a glass ves

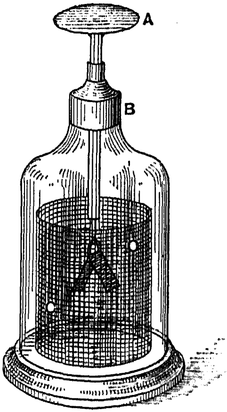


Fig. 1

a metal rod, with a di
and terminating belov
leaf, passes through t
the rod passing throug
inside and out with s
varnish and fitting tig
mouth of the vessel.

When the gold lea
are repelled from each
amount of the divergen
tion of the degree of
desirable to protect th
influence of electrifie
happen to be near the
any electrification there may be on the surfac
this we take advantage of the property of ele
in Art. 33), that a closed metallic vessel comp
inside it from the electrical action of bodies
gold leaves could be completely surrounded b
would be perfectly shielded from extraneous el
however is not practicable, as the metal cas
leaves from observation. In practice, sufficient
by a cylinder of metal gauze connected to eart
Fig. 1, care being taken that the top of the
above the gold leaves.

If the disc of the electroscope is touched b

fied body, will have the same kind of electrification as will repel each other. This repulsion will cease as soon fied body is removed.

If, when the electrified body is near the electrosc connected to the ground by a metal wire, then the me troscope, the wire and the ground, will correspond to piece of metal in the experiment described in Art. 3. T the body to be negatively electrified, the positive elec be on the disc, while the negative will go to the mos of the system consisting of the metal of the electros and the ground, i.e. the negative electrification will go and the gold leaves will be free from electrification. T to repel each other and remain closed. If the wire is the disc while the electrified body remains in the neigh gold leaves will remain closed as long as the electrified stationary, but if this is removed far away from the el gold leaves diverge. The positive electrification, wh electrified body was close to the electroscope, concent the disc so as to be as near the electrified body as poss body is removed spreads to the gold leaves and causes th

If, when the electroscope is charged, we wish to dete the charge is positive or negative, all we have to d near to the disc of the electroscope a stick of sealing-w been negatively electrified by friction with flannel; th the negatively electrified wax, in consequence of the ind increases the negative electrification on the gold leaves presence of the sealing-wax increases the divergence the original electrification was negative, but if it d divergence the original electrification was positive.

in the vessel. If two or more electrified bodies
the divergence of the gold leaves is the same
bodies are moved about re
or to the vessel. The di
leaves thus measures some
fied body which remains
body is moved about wit
property is called the charge
bodies, A and B, have eq
divergence of the gold leave
is inside the vessel placed on
scope and B far away, as w
far away. A and B are
suspended by dry silk thre
do not allow the electricity
see Art. 6. Again, the charge
that on A if, when C is

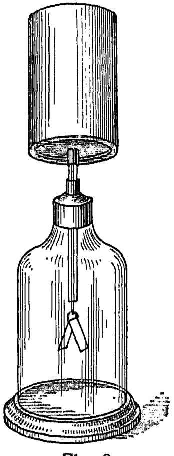


Fig. 2

vessel, it produces the same effect on the ele
duced by A and B when introduced together.
charge has been proved equal to that on A in t
Proceeding in this way we can test what mult
given electrified body is of the charge on anot
take the latter charge as the unit charge we c
in terms of this unit.

Two bodies have equal and opposite charg
simultaneously into the metal vessel they pr
divergence of the gold leaves.

6. Insulators and Conductors. Intr
described in the preceding experiment an elect

ndrawn from the ball. Thus the pith ball loses its charge w ched with a metal wire, though not when touched with a piec ing-wax. We may thus divide bodies into two classes, (1) th ch, when placed in contact with a charged body can discha electrification, these are called *conductors*; (2) those which can harge the electrification of a charged body with which they ontact, these are called *insulators*. The metals, the human bo tions of salts or acids are examples of conductors, while the silk threads, dry glass, ebonite, sulphur, paraffin wax, seal , shellac are examples of insulators.

When a body is entirely surrounded by insulators it is said t *ulated*.

7. *When electrification is excited by friction or by any other pro l charges of positive and negative electricity are always produced.* w this, when the electrification is excited by friction, take a p ealing-wax and electrify it by friction with a piece of flannel; t ugh both the wax and the flannel are charged with electric will, if introduced together into the metal vessel on the dis electroscope (Art. 5), produce no effect on the electroscope, wing that the charge of negative electricity on the wax is e he charge of positive electricity on the flannel. This can be sh more striking way by working a frictional electrical mach ulated and placed inside a large insulated metal vessel in met nexion with the disc of an electroscope; then, although the orous electrical effects can be observed near the machine in vessel, the leaves of the electroscope remain unaffected, show t the total charge inside the vessel connected with the disc been altered though the machine has been in action.

To show that, when a body is electrified by induction, e

electrified by induction, so that its charge of p
excited by this process is equal to its charge of ne

Again, when two charged bodies are conne
the sum of the charges on the bodies is unalter
positive electrification gained by one is equal to
tive electrification lost by the other. To show t
fied metallic bodies, A and B, suspended from si
duce A into the metal vessel, noting the divergen
then introduce B into the vessel and observe the
two bodies are in the vessel together: now take a
round one end of a dry glass rod and, holding the
place the wire so that it is in contact with A an
no alteration in the divergence of the gold lea
by this process, showing that the sum of the ch
unaltered. Take away the wire and remove B
now again observe the divergence of the gol
(except in very special cases) be the same as it w
into the vessel, thus proving that, though a tra
cation between A and B has taken place, the su
A and B has not changed.

8. Force between bodies charged
*When two charged bodies are at a distance r apa
compared with the greatest linear dimension of ei
repulsion between them is proportional to the pro
and inversely proportional to the square of the di*

This law was first proved by Coulomb by di
the force between electrified bodies; there a
methods by which the law can be much more rig
as these can be most conveniently considered

The unit charge of electricity is defined to be such that when
ies each have this charge, and are separated by unit distanc
they are repelled from each other with unit force. The dimens
he charged bodies are assumed to be very small compared
unit distance.

It follows from this definition and the law of force previo
nciated that the repulsion between two small bodies with cha
d e' placed in air at a distance r apart is equal to

$$\frac{ee'}{r^2}.$$

The expression ee'/r^2 will express the force between two cha
ies, whatever the signs of their electrifications, if we agree t
n the expression is positive, it indicates that the force betv
bodies is a repulsion, and that, when this expression is nega
dicates that the force is an attraction. When the charges on
ies are of the same kind ee' is positive, the force is then repuls
n the charges are of opposite sign ee' is negative, the force betw
bodies is then attractive.

Electric Intensity. The electric intensity at any poin
force acting on a small body charged with unit positive ch
n placed at the point, the electrification of the rest of the sys
g supposed to be undisturbed by the presence of this
rge.

Total Normal Electric Induction over a Surf
gine a surface drawn anywhere in the electric field, and let
ace be completely divided up as
the figure, into a network of
hes, each mesh being so small

meshes on the surface is defined to be the tota duction over the surface. This is algebraicall relation

$$I = \Sigma N\omega,$$

where I is the total normal electric induction, N t electric intensity resolved along the normal dra of the surface at a point in a mesh, and ω is the a symbol Σ denotes that the sum of the products N all the meshes drawn on the surface.

With the notation of the integral calculus

$$I = \int N dS,$$

where dS is an element of the surface, the integ over the surface.

10. Gauss's Theorem. We can prove a about the forces between electrified bodies, whi in the following discussion of Electrostatics, by t due to Gauss. This theorem may be stated thu *electric induction over any closed surface drawn i equal to 4π times the total charge of electricity insid*

We shall first prove this theorem when the ele to a single charged body.

Let O (Fig. 4) be the charged body, whose d posed to be so small, compared with its distanc

at which the electric intensity is measured, that it m
as a point. Let e be the charge on this body.

Let $PQRS$ be one of the small meshes drawn on t
area being so small that $PQRS$ may be regarded as pla
Q, R, S, and let a plane through R at right angles to $($
OP respectively in u, v, w: with centre O describe a
radius, and let the lines OP, OQ, OR, OS cut the surfac
in the points p, q, r, s respectively. The area $PQRS$ is
so small that the electric intensity may be regarded a
it; we may take as the value of the electric intensity
the value it has at R.

The contribution of this mesh to the total norma
by definition, equal to

$$\text{area } PQRS \times N,$$

where N is the normal component of the electric inte

Now $$N = \frac{e}{OR^2} \cos\theta,$$

where θ is the angle between the outward normal to t
and OR the direction of the electric intensity. The
surface is at right angles to $PQRS$, and OR is at righ
area $Ruvw$, and hence the angle between the normal
and OR is equal to the angles between the planes PQ

Hence

$$\text{area } PQRS \times \cos\theta = \text{the area of the projectio}$$
$$\text{area } PQRS \text{ on the}$$
$$= \text{area } Ruvw \quad \ldots\ldots\ldots\ldots$$

Consider the figures $Ruvw$ and $rspq$. Ru is parallel
are in the same plane and both at right angles to OR,
reasons Rv is parallel to rq, vw to pq, uw to sp. Th

The contribution of the mesh $PQRS$ to the to
is equal to

$$\text{area } PQRS \times \frac{e}{OR^2} \times \cos\theta$$

$$= e \times \frac{\text{area } Ruvw}{OR^2} \text{ by equation}$$

$$= e \times \text{area } pqrs \text{ by equation (}$$

Thus the contribution of the mesh to the tot
is equal to e times the area cut off a sphere of
centre at O by a cone having the mesh for a base

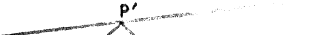
By dividing up any finite portion of the surfa
taking the sum of the contributions of each me
total normal induction over the surface is equal
cut off a sphere of unit radius with its centre at
the boundary of the surface as base and its vert

Let us now apply the results we have obtai
closed surface.

First take the case where O is inside the surface
induction over the surface is equal to e times th
cut off the unit sphere by cones with their bases
their vertices at O, and since the meshes complete
surface the sum of the areas cut off the unit sphe
be the area of the sphere, which is equal to 4π, sinc
Thus the total normal induction over the closed

Next consider the case when O is outside the

Draw a cone with its vertex at O cutting the c
areas $PQRS$, $P'Q'R'S'$. Then the magnitude of t



$PQRS$ the electric intensity points along the outward so that the sign of the component resolved along the c normal is positive; while over the surface $P'Q'R'S'$ t tensity is in the direction of the inward drawn normal s of its component along the outward drawn normal is n the total normal induction over $PQRS$ is of opposite si $P'Q'R'S'$, and since they are equal in magnitude they other as far as the total normal induction is concern whole of the closed surface can be divided up in this with their vertices at O, and since the two sections of cones neutralize each other, the total normal induction surface will be zero.

We thus see that when the electric field is due to a s a charge e, the total normal induction over any clos closing the charge is $4\pi e$, while it is equal to zero o surface not enclosing the charge. We have therefore theorem when the field is due to a single small electri

We can easily extend it to the general case when to any distribution of electrification. For we may arising from a number of small bodies having charges Let N be the component along the outward drawn surface of the resultant electric intensity, N_1 the con same direction due to e_1, N_2 that due to e_2 and so on;

$$N = N_1 + N_2 + N_3 + \dots.$$

If ω is the area of the mesh at which the normal el is N, the total normal induction over the surface is

$$\begin{aligned} &\Sigma N\omega \\ &= \Sigma (N_1 + N_2 + N_3 + \dots)\,\omega \\ &= \Sigma N_1\omega + \Sigma N_2\omega + \Sigma N_3\omega + \dots, \end{aligned}$$

several charged bodies, i.e. that due to the actual charge of electricity inside the closed surface ov induction is taken.

11. Electric intensity at a point outs charged sphere. Let us now apply the th electric intensity at any point in the region outsid charged with electricity.

Let O be the centre of the sphere, P a point o which the electric intensity is required.

Through P draw a spherical surface with its be the electric intensity at P. Since the charged electrified, the direction of the intensity will be the same value R at any point on the spherical Hence since at each point on this surface the tensity is equal to R, the total normal inducti through P is equal to $R \times$ (surface of the sphere By Gauss's theorem this is equal to 4π times the the spherical surface, that is to 4π times the sphere. If e is this charge we have therefore

$$R \times 4\pi OP^2 = 4\pi e,$$

$$R = \frac{e}{OP^2}.$$

Hence the intensity at a point outside a uniforml is the same as if the charge on the sphere were centre.

12. Electric intensity at a point ins electrified spherical shell. Let Q be a poin R the electric intensity at that point. Through surface, centre O; then as before, the normal ele

ence the electric intensity vanishes at any point inside a uniforn
ectrified spherical shell.

13. Infinite Cylinder uniformly electrified. We shall n
nsider the case of an infinitely long circular cylinder uniforn
ectrified. Let P be a point outside the cylinder at which we w
find the electric intensity. Through P describe a circular cylin
axial with the electrified one, draw two planes at right angles to
is of the cylinder at unit distance apart, and consider the to
rmal induction over the closed surface formed by the curved s
ce of the cylinder through P and the two plane ends. Since
ectrified cylinder is infinitely long and is symmetrical about its a
e electric intensity at all points at the same distance from the a
the cylinder will be the same, and the electric intensity at P will
mmetry be along a radius drawn through P at right angles to
is of the cylinder.

Thus the electric intensity at any point on either plane end of
linder will be in the plane of that end, and will therefore have
mponent at right angles to it; the plane ends will therefore c
bute nothing to the total normal induction over the surface.
ch point of the cylindrical surface the electric intensity is at rig
gles to the curved surface and is equal to R. The total norn
luction over the surface is therefore

$$R \times \text{(area of the curved surface of the cylinder)}.$$

But since the length of the curved surface is unity its area is eq
$2\pi r$, where r is the distance of P from the axis of the cylinder. I
the charge per unit length on the electrified cylinder, then
uss's theorem the total normal induction over the surface is eq
$4\pi E$. The total normal induction is however equal to $R \times 2$
nce

$$R \times 2\pi r = 4\pi E,$$

14. Uniformly electrified infinite plane ...
ee by symmetry (1) that the electric intensity will
plane, (2) that the electric intensity will be cons

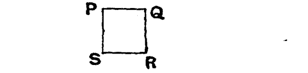


Fig. 6

in a plane parallel
one. Draw a cylin
the axis of the c
right angles to th
of the cylinder being planes at right angles to
this cylinder encloses no electrification the tot
tion over its surface is zero by Gauss's theorem
electric intensity is parallel to the axis of the cyl
intensity vanishes over the curved surface of the cy
the electric intensity at a point on the face *PQ*—
outward drawn normal if the electrification on the p
F′ the electric intensity at a point on the face *RS*, ω
of the faces *PQ* or *RS*, then the total normal in
surface *PQRS* is equal to

$$F\omega - F'\omega;$$

and since this vanishes by Gauss's theorem

$$F = F',$$

or the electric intensity at any point, due to the i
charged plane, is independent of the distance of th
plane. It is, therefore, constant in magnitude at
field, acting upwards in the region above the plan
the region below it.

To find the magnitude of the intensity at *P*.
(Fig. 7) a line at right angles to the plane and prolo
Q is as far below

agnitude of the electric intensity at any point in the field, ω t
ea of either of the flat ends of the cylindrical surface. Th
e part of the total normal induction over the surface $PQRS$ d
the flat end through P is $R\omega$. The part due to the flat end throu
will also be equal to this and will be of the same sign, since t
tensity at Q is along the outward drawn normal. Thus since t
rmal intensity vanishes over the curved surface of $PQRS$ the to
rmal induction over the closed surface is $2R\omega$. If σ is the quant
electricity per unit area of the plane the charge of electricity insi
e closed surface is $\sigma\omega$; hence by Gauss's theorem

$$2R\omega = 4\pi\sigma\omega,$$

$$R = 2\pi\sigma.$$

y comparing this with the results given in Arts. 11 and 13 t
udent may easily prove that the intensity due to the charged pla
rface is half that just outside a charged spherical or cylindri
rface having the same charge of electricity per unit area.

15. Lines of Force. A line of force is a curve drawn in t
ectric field, such that its tangent at any point is parallel to t
ectric intensity at that point.

16. Electric Potential. This is defined as follows: t
ectric potential at a point P exceeds that at Q by the work do
the electric field on a body charged with unit
electricity when the latter passes from P to
The path by which the unit of electricity
avels from P to Q is immaterial, as the work
ne will be the same whatever the nature of
e path. To prove this, suppose that the
ork done on the unit charge when it travels

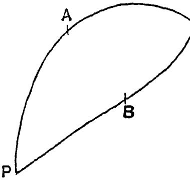


Fig. 8

applied forces in bringing it back from Q to P QBP. Thus though the unit of electricity is back which it started, and if the field is entirely du electricity, everything is the same as when it sta our hypothesis is correct, gained work. This is n with the principle of the Conservation of Energy, conclude that the hypothesis on which it is found work done on unit electric charge when it travels pends on the path by which it travels, is incorrec

Since electric phenomena only depend upon diff tial it is immaterial what point we take as the one the potential zero. In mathematical investigations expression for the potential to assume as the point one at an infinite distance from all the electrified bodi

If P and Q are two points so near together tha tensity may be regarded as constant over the distan work done by the field on unit charge when it trav is $F \times PQ$, if F is the electric intensity resolved in t If V_P, V_Q denote the potentials at P and Q respect by definition $V_P - V_Q$ is the work done by the field when it goes from P to Q we have

$$V_P - V_Q = F \times PQ,$$

hence

$$F = \frac{V_P - V_Q}{PQ} \quad \ldots\ldots\ldots\ldots$$

thus the electric intensity in any direction is equa diminution of the potential in that direction.

Hence if we draw a surface such that the poten over the surface (a surface of this kind is called a surface) the electric intensity at any point on the

The surface of a conductor placed in an electric field must be ipotential surface when the field is in equilibrium, for there no tangential electric intensity, otherwise the electricity on face would move along the surface and there could not be equ um. It is this fact that makes the conception of the potentia ortant in electrostatics, for the surfaces of all bodies mad al are equipotential surfaces.

A positively charged conductor does not necessarily have a p potential, as the potential depends on *all* the charges in the fi

17. Potential due to a uniformly charged sphere. ential at P is the work done by the electric field when unit ch aken from P to an infinite ance. Let us suppose that the charge travels from P to an nite distance along a straight

Fig. 9

passing through the centre of the sphere. Let $QRST$ be a s points very near together along this line. If e is the charge on ere, O its centre, the electric intensity at Q is e/OQ^2, while tha s e/OR^2; as Q and R are very near together these quantities y nearly equal, and we may take the average electric inten ween Q and R as equal to $e/OQ \, . \, OR$, the geometric mean of ensities at Q and R. Hence the work done by the field as the rge goes from Q to R is equal to

$$\frac{e}{OQ \, . \, OR} QR$$

$$\frac{e}{OQ \, . \, OR} (OR - OQ)$$

$$\frac{e}{} \quad \frac{e}{}$$

and so on. The work done by the field as the charg
is the sum of these expressions, and this sum is equ

$$\frac{e}{OQ} - \frac{e}{OT},$$

and we see, by dividing up the distance betweer
number of small intervals and repeating the abov
expression will be true when Q and T are a fin
and that it always represents the work done by tl
charge as long as Q and T are two points on a ra
The potential at P is the work done by the field wl
goes from P to an infinite distance, and is therefor
result equal to

$$\frac{e}{OP}.$$

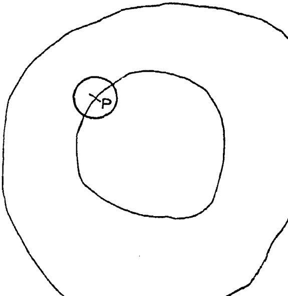
This is also evidently the potential at P of a charge
body enclosing O if the dimensions of the body ove
is spread are infinitesimal in comparison with OP.

**18. The electric intensity vanishes at a
a closed equipotential surface which does
electric charge.** We shall first prove that the
stant throughout the volume enclosed by the su
follow by equation (1), Art. 16, that the electric
throughout this volume.

For if the potential is not constant it will be
series of equipotential surfaces inside the given o
the equipotential surface for which the potential
not quite, the same as for the given surface. A
potential between this and the outer surface is v
surfaces will be close together, and they cannot

at the normal electric intensity over the second surface is eve
ere in the direction of the outward drawn normal to the surfa
d therefore that the total normal electric induction over the surf
l be *positive*. Hence there must be a positive charge inside
rface, as the total normal induction over the surface is, by Gaus
eorem, proportional to the charge enclosed by the surface. Hen
by hypothesis there is no charge inside the surface, we see t
e potential over the inner surface cannot be greater than that
e outer surface. If the potential at the inner surface were
an that at the outer, then the normal electric intensity would
erywhere in the direction of the inward normal, and, as befo
can show by Gauss's theorem that this would require a negat
arge inside the surface. Hence, as there is no charge either posit
negative the potential at the inner surface can neither be grea
r less than at the outer surface, and must therefore be equal to
this way we see that the potential at all points inside the surf
ust have the same value as at the surface, and since the poten
constant the electric intensity will vanish inside the surface.

19. It follows from this that if we have a closed hollow conduc
ere will be no electrification on its inner surface unless there
ctrified bodies inside the hollow.
t Fig. 10 represent the conductor
th a cavity inside it. To prove that
ere is no electrification at P a point
the inner surface, take any closed
rface enclosing a small portion α of
e inner surface near P; by Gauss's
eorem the charge on α is propor-
nal to the total normal electric in-



cavity being the surface of a conductor is an equ and as we have just seen the electric intensity ins is zero unless it encloses electric charges. Thus si tensity vanishes at each point on the closed surfa the charge at a must vanish; in this way we can s electrification at any point on the inner cavity. is all on the outer surface of the conductor.

20. Cavendish Experiment. The result that when the force between two charged bodie as the square of the distance between them the vanishes throughout the interior of an electrified co no charge, leads to the most rigorous experime truth of this law.

Let us for simplicity confine our attention to electrified conductor is a sphere positively electrifi

Consider the state of things at a point P insic centre is O, Fig. 11: through P draw a plane at ri

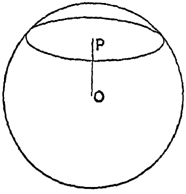


Fig. 11

The electrification on the port above this plane produces an el the direction PO, while the elec portion of the sphere below th an electric intensity in the dire the law of force is that of the in two intensities balance each o distance from P of the electrif plane being compensated by the larger electrified a

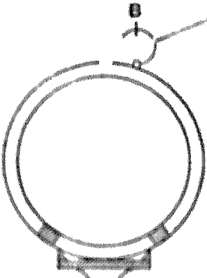
Now suppose that intensity varies as r^{-p}, then i 2 the intensity diminishes more quickly as the distar

the increased distance when the law of force is that of the inv
uare, it will be more than sufficient to do so when p is less tha
this case the electrification below the plane will gain the u
nd, and the electric intensity at P will be in the direction OP.

Now suppose we have two concentric metal spheres connecte
ire, and that we electrify the outer sphere positively, then if p
re will be no electric intensity inside the outer sphere, and th
no movement of electricity to the inner sphere which will th
remain unelectrified. If p is greater than 2 we have seen
electric intensity due to the positive charge on the outer sp
be towards the centre of the sphere, i.e. the force on a nega
rge will be from the inner sphere towards the outer. Nega
tricity will therefore flow from the inner sphere, which will be
h a positive charge.

If however p is less than 2, the electric intensity due to the ch
the outer sphere will be from the centre of the sphere, and
ection of the force acting on a positive charge will be from
er sphere to the outer. Positive electricity will therefore flow f
inner sphere to the outer, so that the inner sphere will be
h a negative charge.

Thus, according as p is greater than, equal to or less than 2,
rge on the inner sphere will be positive,
or negative. By testing the state of
trification on the inner sphere we can
refore test the law of force. This is what
done by Cavendish in an experiment
de by him, and which goes by his name*.
following is a description of a slight
dification, due to Maxwell, of Cavendish's
inal experiment.



of an ebonite ring. Connection between
spheres is made by a wire fastened to
which acts as a lid to a small hole in the o
wire and the disc are lifted up by a si
condition of the inner sphere can be tested b
wire connected to an electroscope (or pre
electrometer, see Art. 60) through the hol
with the inner sphere. The experiment is
the two spheres are in connection a charg
municated to the outer sphere, the connecti
is then broken by lifting the disc by means
outer sphere is then discharged and kept c
testing wire is then introduced through the h
with the inner sphere. Not the slightest ef
can be detected, showing that if there is a
sphere it is too small to affect the electrosc
sensitiveness of the electroscope or electrom
suspended by a silk thread, is placed at a co
the two spheres. After the outer sphere is char
the brass ball is touched and then left insula
gets by induction a negative charge amount
tion, say α, of the original charge communica
Now when the outer sphere is connected to ea
on the ball will induce a positive charge on t
a calculable fraction, say β, of the charge on t
the outer sphere from the earth and dischar
charge on the outer sphere will be free to g
this is connected to the sphere. When the b
from the sphere this charge is sufficient to d

vendish experiment that this charge is less than $\alpha\beta$ of the origi
arge, we can calculate that p must differ from 2 by less than a c
n quantity. In this way it has been shown that p differs from 2
s than 1/20,000.

21. Definition of surface density. When the electrificati
confined to the surface of a body, the charge per unit area is cal
e surface density of the electricity.

22. Coulomb's Law. The electric intensity at a point
se to the surface of a conductor surrounded by air is at right ang
the surface and is equal to $4\pi\sigma$, where σ is the surface density of
ctrification.

The first part of this law follows from Art. 16, since the surfa
a conductor is an equipotential surface. To prove the second p
ke on the surface a small area around P
g. 13) and through the boundary of this
ea draw the cylinder whose generating
es are parallel to the normal at P. Let
s cylinder be truncated at T and S by
nes parallel to the tangent plane at P.

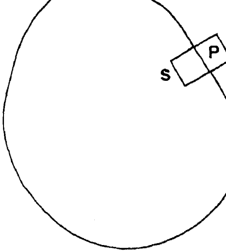


Fig. 13

The total normal electric induction over
s cylinder is $R\omega$, where R is the normal
ctric intensity and ω the area of the cross
tion. For $R\omega$ is the part of the total
rmal induction due to the end T of the cylinder, and this is
ly part of the surface of the cylinder which contributes anyth
the total normal induction. For the intensity along that part
e curved surface of the cylinder which is in air is tangential to
rface and therefore has no component along the normal, wh
ce the electric intensity vanishes inside the conductor the p
the surface which is inside the conductor will not contrib
ything to the total induction. If σ is the surface density of
ctricity at P the charge inside the cylinder is $\omega\sigma$; hence by Gaus

The result expressed by this equation is k
Law. It requires modification when the conduct
by air, but by some other insulator. See Art.

23. Energy of an electrified system.
ductors are placed in an electric field, and if E_1
first conductor, V_1 its potential, E_2 the charge o
tor, V_2 its potential, and so on, then we can sh
energy of this system of conductors is equal to

$$\tfrac{1}{2}(E_1V_1 + E_2V_2 + \ldots).$$

To prove this we notice that the potentials
depend upon the charges of electricity on the
way that if the charge on every part of the s
times, the potential at every point in the system
m times.

To find the energy of the system of conduc
that each conductor is originally uncharged,
and that we bring a charge E_1/n from an infini
conductor, a charge E_2/n from an infinite dista
ductor, a charge E_3/n to the third conductor.
has been done, the potential of the first condu
of the second V_2/n, and so on. Let us call thi
operation. Then bring from an infinite distan
first conductor, E_2/n to the second, and so on
done the potentials of the conductors will be
this the second stage of the operation. Repeat
first conductor has the charge E_1 and the po
conductor the charge E_2 and the potential V_2

Then in the first stage the potential of the
at the beginning, and V_1/n at the end; the wo

In the second stage of the operations the potential of the
nductor is V_1/n at the beginning, and $2V_1/n$ at the end, so tha
rk spent in bringing up the charge E_1/n to the first conduct
ater than $\frac{E_1}{n}\cdot\frac{V_1}{n}$ but less than $\frac{E_1}{n}\cdot\frac{2V_1}{n}$; similarly the work s
bringing up the charge E_2/n to the second conductor is greater
$\cdot\frac{V_2}{n}$ but less than $\frac{E_2}{n}\cdot\frac{2V_2}{n}$. Thus if ${}_2Q_1$ is the work spent in
ge in charging the first conductor we have

$$ {}_2Q_1 > \frac{1}{n^2}E_1V_1, \quad {}_2Q_1 < \frac{2}{n^2}E_1V_1. $$

Similarly if ${}_3Q_1$ is the work spent in the third stage in charging
t conductor we have

$$ {}_3Q_1 > \frac{2}{n^2}E_1V_1, \quad {}_3Q_1 < \frac{3}{n^2}E_1 $$

${}_nQ_1$ the work spent in the last stage, is

$$ > \frac{n-1}{n^2}E_1V_1, $$

$$ < \frac{n}{n^2}E_1V_1. $$

Now Q_1 the total amount of work spent in charging the first c
or is equal to ${}_1Q_1 + {}_2Q_1 + \ldots + {}_nQ_1$, and is therefore

$$ > \frac{1+2+3+\ldots+(n-1)}{n^2}E_1V_1, \quad < \frac{1+2+3+\ldots+n}{n^2}E_1V_1, $$

$$ > \frac{(n-1)n}{2n^2}E_1V_1, \quad < \frac{n(n+1)}{2n^2}E_1V_1, $$

$$ > \frac{1}{2}\left(1-\frac{1}{n}\right)E_1V_1, \quad < \frac{1}{2}\left(1+\frac{1}{n}\right)E_1V_1 $$

The work done in charging the conductors is system as electrical energy, the potential energy of equal to the work done in charging up the system depends on the final state of the system and is in way that state is arrived at. Hence we see from that the energy of a system of conductors is one ha products obtained by multiplying the charge of each potential.

24. Relation between the potentials a the conductors. Superposition of electric V' be the potential at any point P when the first charge E_1 and all the other conductors are withou the potential at P when the second conductor has t all the other conductors are without charge; the conductor has the charge E_1, the second the charg other conductors are without charge, the potent $V' + V''$.

The conditions to be satisfied in this case are on the conductors should have the given values and of the conductors should be equipotential surfaces.

Now consider the distribution of electrificatio conductor has the charge E_1 and the rest are wit satisfies the conditions that the conductors are e faces, that the charge on the first conductor is charges on the other conductors are zero. The distri cation when the second conductor is charged and t satisfies the conditions that the conductors are equi that the charge on the first conductor is zero, that

st of the conductors are uncharged. But when two systems ectrification are superposed, the potential at P is the sum of t tentials due to the two systems separately, i.e. the potential is $V' + V''$, and hence the theorem is true.

25. We can extend this reasoning to the general case in which the potential at P when the first conductor has the charge E e other conductors being uncharged, V'' the potential at P wh e second conductor has the charge E_2, the other conductors bei charged, V''' the potential at P when the charge on the third co ictor is E_3, the other conductors being uncharged, and so on; a e then see that when the first conductor has the charge E_1, t cond the charge E_2, the third the charge E_3, and so on, the potent P is

$$V' + V'' + V''' + \ldots.$$

26. When the first conductor has the charge E_1, the other co ictors being uncharged and insulated, the potentials of the co ictors will be proportional to E_1, that is, the potentials of the fir cond, third, etc. conductors will be respectively

$$p_{11}E_1, \quad p_{12}E_1, \quad p_{13}E_1, \ldots,$$

here p_{11}, p_{12}, p_{13} are quantities which do not depend upon t iarges of the conductors or their potentials, but only upon th iapes and sizes and their positions with reference to each oth he quantities p_{11}, p_{12}, p_{13}, etc. are called *coefficients of potential*; th operties are further considered in Arts. 27–31. When the seco nductor has the charge E_2, the other conductors being uncharg id insulated, the potentials of the conductors will be proportio E_2, and the potentials of the first, second, third, etc. conduct ill be

$$p_{21}E_2, \quad p_{22}E_2, \quad p_{23}E_2, \ldots.$$

so on, V_1 the potential of the first conductor will
equation

$$V_1 = p_{11}E_1 + p_{21}E_2 + p_{31}E_3 + \dots,$$

V_2 the potential of the second conductor by the equ

$$V_2 = p_{12}E_1 + p_{22}E_2 + p_{32}E_3 + \dots;$$

if V_3 is the potential of the third conductor

$$V_3 = p_{13}E_1 + p_{23}E_2 + p_{33}E_3 + \dots,$$

....

If we solve these equations we get

$$E_1 = q_{11}V_1 + q_{21}V_2 + q_{31}V_3 + \dots,$$

$$E_2 = q_{12}V_1 + q_{22}V_2 + q_{32}V_3 + \dots,$$

where the q's are functions of the p's and only de
configuration of the system of conductors. The q'
efficients of capacity when the two suffixes are the sam
of induction when the suffixes are different.

27. We shall now show that the coefficients whic
equations are not all independent, but that

$$p_{21} = p_{12}.$$

To prove this let us suppose that only the first a
ductors have any charges, the others being without
sulated. Then we may imagine the system charged, b
up the charge E_1 from an infinite distance to the first
leaving all the other conductors uncharged, and then
been done, bringing up the charge E_2 from an infinite
second conductor. The work done in bringing the c
the first conductor will be the energy of the system

Hence the work done in bringing up the first instalme
between

$$p_{12}E_1\frac{E_2}{n} \text{ and } \left(p_{12}E_1 + p_{22}\frac{E_2}{n}\right)\frac{E_2}{n}.$$

Similarly the work done in bringing up the second i
E_2/n will be between

$$\left(p_{12}E_1 + p_{22}\frac{E_2}{n}\right)\frac{E_2}{n} \text{ and } \left(p_{12}E_1 + p_{22}\frac{2E_2}{n}\right)\frac{E_2}{n},$$

and the work done in bringing up the last instalment of t
will be between

$$\left(p_{12}E_1 + p_{22}\frac{(n-1)E_2}{n}\right)\frac{E_2}{n} \text{ and } \left(p_{12}E_1 + p_{22}\frac{nE_2}{n}\right)\frac{E_2}{n}$$

Thus the total amount of work done in bringing up the
will be between

$$p_{12}E_1E_2 + \frac{1+2+3\dots+(n-1)}{n^2}p_{22}E_2^2$$

and $$p_{12}E_1E_2 + \frac{1+2+3\dots+n}{n^2}p_{22}E_2^2,$$

that is, between

$$p_{12}E_1E_2 + \frac{1}{2}\left(1-\frac{1}{n}\right)p_{22}E_2^2 \text{ and } p_{12}E_1E_2 + \frac{1}{2}\left(1+\frac{1}{n}\right)p_{22}E_2^2$$

but if n is very great these two expressions become equal to

$$p_{12}E_1E_2 + \tfrac{1}{2}p_{22}E_2^2,$$

which is therefore the work done in bringing up the charge
second conductor when the first conductor has already rece
charge E_1. Hence the work done in bringing up first the c
and then E_2 is

$$\tfrac{1}{2}p_{11}E_1^2 + p_{12}E_1E_2 + \tfrac{1}{2}p_{22}E_2^2$$

It follows from the way in which the q's can be expressed in terms of the p's, that $q_{21} = q_{12}$.

28. Now p_{12} is the potential of the second conductor when unit charge is given to the first, the other conductors being insulated and without charge, and p_{21} is the potential of the first conductor when unit charge is given to the second. But we have just seen that $p_{21} = p_{12}$, hence the potential of the second conductor when insulated and without charge due to unit charge on the first is equal to the potential of the first when insulated and without charge due to unit charge on the second, the remaining conductors being in each case insulated and without charge.

29. Let us consider some examples of this theorem. Let us suppose that the first conductor is a sphere with its centre at O, and that the second conductor is very small and placed at P, then if P is outside the sphere we know by Art. 17 that if unit charge is given to the sphere the potential at P is increased by $1/OP$. It follows from the preceding article that if unit charge be placed at P the potential of the sphere when insulated is increased by $1/OP$.

If P is inside the sphere then when unit charge is given to the sphere the potential at P is increased by $1/a$, where a is the radius of the sphere. Hence if the sphere is insulated and a unit charge placed at P the potential of the sphere is increased by $1/a$. Thus the increase in the potential of the sphere is independent of the position of P as long as it is inside the sphere.

Since the potential inside any closed conductor which does not include any charged bodies is constant, by Art. 18, we see by taking as our first conductor a closed surface, and as our second conductor a small body placed at a point P anywhere inside this surface, that since the potential at P due to unit charge on the conductor is independent of the position of P, the potential of the conductor when insulated due to a charge at P is independent of the position of P. Thus however a charged body is moved about inside a closed insulated conductor the potential of the conductor will remain constant. An example of this is afforded by the experiment described in Art. 5 the deflection of the electroscope is independent of the position of the charged bodies inside the insulated closed conductor.

Again, take the case when the first conductor is charged, the
nsulated and uncharged; then

$$V_1 = p_{11}E_1,$$

$$V_2 = p_{12}E_1,$$

$$\frac{V_1}{V_2} = \frac{p_{11}}{p_{12}}.$$

v suppose that the first conductor is connected to earth while
e E_2 is given to the second conductor, all the other conductors
ncharged; then since $V_1 = 0$ we have

$$0 = p_{11}E_1 + p_{12}E_2,$$

$$\frac{E_2}{E_1} = -\frac{p_{11}}{p_{12}} = -\frac{V_1}{V_2}$$

preceding equation.

nce if a charge be given to the first conductor, all the others
nsulated, the ratio of the potential of the second conductor
of the first will be equal in magnitude but opposite in sign
charge induced on the first conductor, when connected to
by unit charge on the second conductor.

an example of this result, suppose that the first conductor
here with its centre at O, and that the second conductor is
l body at a point P outside the sphere; then if unit charge be
o the sphere, the potential of the body at P is a/OP times the
ial of the sphere, where a is the radius of the sphere; hence, by
eorem of this article, when unit charge is placed at P, and the
is connected to the earth, there will be a negative charge on
here equal to a/OP.

other example of this result is when the first conductor com
surrounds the second; then since the potential inside the first
ctor is constant when all the conductors inside are free from
, the potential of the second conductor when a charge is given
first conductor will be the same as that of the first. Hence
he above result it follows that when the first conductor is
cted to earth, and a charge given to the second, the charge
d on the first conductor will be equal and opposite to that given
second.

Another consequence of this result is that if S be an equipotential surface when the first conductor is charged, all the others being insulated, then if the first conductor be connected to earth the charge induced on it by a charge on a small body P remains the same however P may be moved about, provided that P always keeps on the surface S.

31. As an example in the calculation of coefficients of capacity and induction, we shall take the case when the conductors are two concentric spherical shells. Let a be the radius of the inner shell, which we shall call the first conductor, b the radius of the outer shell, which we shall call the second conductor. Let E_1, E_2 be the charges of electricity on the inner and outer shells respectively, V_1, V_2 the corresponding potentials of these shells.

Then if there were no charge on the outer shell the charge E_1 on the inner would produce a potential E_1/a on its own surface, and a potential E_1/b on the surface of the outer shell; hence, Art. 26,

$$p_{11} = \frac{1}{a}; \quad p_{12} = \frac{1}{b}.$$

The charge E_2 on the outer shell would, if there were no charge on the inner shell, make the potential inside the outer shell constant and equal to the potential at the surface of the outer shell. This potential is equal to E_2/b, so that the potential of the first conductor due to the charge E_2 on the second is E_2/b, which is also equal to the potential of the second conductor due to the charge E_2; hence, by Art. 26,

$$p_{21} = \frac{1}{b}, \quad p_{22} = \frac{1}{b}.$$

We have therefore

$$V_1 = p_{11}E_1 + p_{21}E_2 = \frac{E_1}{a} + \frac{E_2}{b},$$

$$V_2 = p_{12}E_1 + p_{22}E_2 = \frac{E_1}{b} + \frac{E_2}{b}.$$

Solving these equations, we get

$$E_1 = \frac{ab}{b-a} V_1 - \frac{ab}{b-a} V_2,$$

$$E_2 = -\frac{ab}{b-a} V_1 + \frac{b^2}{b-a} V_2.$$

Hence

$$q_{11} = \frac{ab}{b-a}, \quad q_{12} = q_{21} = -\frac{ab}{b-a}, \quad q_{22} = \frac{b^2}{b-a}.$$

We notice that q_{12} is negative; this, as we shall prove later, is always true whatever the shape and position of the two conductors.

32. Another case we shall consider is that of two spheres the distance between whose centres is very large compared with the radius of either. Let a be the radius of the first sphere, b that of the second, R the distance between their centres, E_1, E_2 the charges, V_1, V_2 the potentials of the two spheres. Then if there were no charge on the second sphere, the potential at the surface of the first sphere would, if the distance between the spheres were very great, be approximately E_1/a, while the potential of the second sphere would be approximately E_1/R; hence

$$p_{11} = \frac{1}{a}, \quad p_{12} = \frac{1}{R},$$

approximately.

Similarly, if there were no charge on the first sphere, but a charge E_2 on the second, the potential of the first sphere would be E_2/R, that of the second E_2/b, approximately; hence we have approximately

$$p_{21} = \frac{1}{R}, \quad p_{22} = \frac{1}{b}.$$

So that approximately

$$V_1 = \frac{E_1}{a} + \frac{E_2}{R},$$

$$V_2 = \frac{E_1}{R} + \frac{E_2}{b}.$$

Solving these equations we get

$$E_1 = \frac{aR^2}{R^2 - ab} V_1 - \frac{abR}{R^2 - ab} V_2,$$

$$E_2 = -\frac{abR}{R^2 - ab} V_1 + \frac{bR^2}{R^2 - ab} V_2,$$

hence when R is large compared with both a and b

$$q_{11} = \frac{aR^2}{R^2 - ab}, \quad q_{12} = q_{21} = -\frac{abR}{R^2 - ab}, \quad q_{22} = \frac{bR^2}{R^2 - ab},$$

approximately.

We see that as before q_{12} is negative. We also notice tha
q_{12} become larger the nearer the spheres are together.

33. Electric Screens. As an example of the use of coe
of capacity we shall consider the case of three conductors,
and shall suppose that the first of these conductors A is, as ir
inside the third conductor C, which is supposed to be a closec

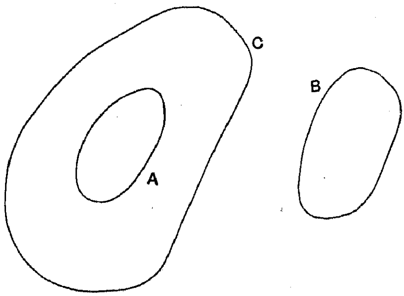


Fig. 14

while the second conductor B is outside C. Then if E_1, V
E_3, V_3 denote the charges and potentials of the conducto
respectively, q_{11}, q_{22}, ... q_{12}, ... the coefficients of capacity
tion, we have

$$E_1 = q_{11}V_1 + q_{12}V_2 + q_{13}V_3 \quad \cdots\cdots\cdots$$
$$E_2 = q_{12}V_1 + q_{22}V_2 + q_{23}V_3 \quad \cdots\cdots\cdots$$
$$E_3 = q_{13}V_1 + q_{23}V_2 + q_{33}V_3 \quad \cdots\cdots\cdots$$

Now let us suppose that the conductor C is connected
that V_3 is zero; then, since the potential inside a closed c
constant if it contains no charge, we see that if E_1 is zer
vanish whatever may be the value of V_2. Hence it fo
equation (1) that q_{12} must vanish; putting q_{12} and V_3 be
see from (1) that

$$E_1 = q_{11}V_1,$$

and from (2)

$$E_2 = q_{22}V_2.$$

Thus, in this case, the charge on A if its potential is g
potential if its charge is given, is entirely independent of
that is a charge on B produces no electrical effect on
charge on A produces no electrical effect on B. Thus the
between A and B is entirely cut off by the interposition
conductor at potential zero.

C is called an electric screen since it screens off from A all the effects that might be produced by B. This property of a closed metallic surface at zero potential has very important applications, as it enables us by surrounding our instruments by a metal covering connected with earth to get rid entirely of any electrical effects arising from charged bodies not under our control. Thus, in the experiment described in Art. 4, the gold leaves of the electroscope were protected from the action of external electrified bodies by enclosing them in a surface made of wire-gauze and connected with the earth.

34. Expression for the change in the energy of the system. The energy of the system Q is, by Art. 23, equal to $\frac{1}{2}\Sigma EV$; hence we have, by Art. 27,

$$Q = \tfrac{1}{2}p_{11}E_1^2 + \tfrac{1}{2}p_{22}E_2^2 + \dots p_{12}E_1E_2 + \dots.$$

If the charges are increased to E_1', E_2', etc. the energy Q' corresponding to these charges is given by the equation

$$Q' = \tfrac{1}{2}p_{11}E_1'^2 + \tfrac{1}{2}p_{22}E_2'^2 + \dots p_{12}E_1'E_2' + \dots.$$

The work done in increasing the charges is equal to $Q' - Q$. By the preceding equations

$$\begin{aligned} Q' - Q = {} & (E_1' - E_1)\tfrac{1}{2}\{p_{11}(E_1 + E_1') + p_{12}(E_2 + E_2') + \dots\} \\ & + (E_2' - E_2)\tfrac{1}{2}\{p_{12}(E_1 + E_1') + p_{22}(E_2 + E_2') + \dots\} \\ & + \dots\dots \\ = {} & (E_1' - E_1)\tfrac{1}{2}(V_1' + V_1) + (E_2' - E_2)\tfrac{1}{2}(V_2' + V_2) + \dots, \end{aligned}$$

where V_1', V_2', ... are the potentials of the first, second, ... conductors when their charges are E_1', E_2', ...

Thus the work required to increase the charges is equal to the sum of the products of the increase in the charge on each conductor into the mean of the potentials of the conductor before and after the charges are increased.

If we express Q and Q' by Art. 26 in terms of the potentials instead of the charge, we have

$$Q = \tfrac{1}{2}q_{11}V_1^2 + \tfrac{1}{2}q_{22}V_2^2 + q_{12}V_1V_2 + \dots,$$

$$Q' = \tfrac{1}{2}q_{11}V_1'^2 + \tfrac{1}{2}q_{22}V_2'^2 + q_{12}V_1'V_2' + \dots,$$

and we see that

$$Q' - Q = (V_1' - V_1)\tfrac{1}{2}(E_1 + E_1') + \dots.$$

The gain in electric energy when the potentials are constant is

$$\tfrac{1}{2}\{V_1(E_1' - E_1) + V_2(E_2' - E_2) + \ldots\}.$$

The difference between the loss when the charges are constant and the gain when the potentials are constant is thus equal to

$$\tfrac{1}{2}\{(E_1 - E_1')(V_1 - V_1') + \ldots\} + \tfrac{1}{2}\{(E_1V_1 - E_1'V_1') + \ldots\}.$$

Now for the displaced positions of the system E_1, V_1', E_2, V_2', ... are one set of corresponding values of the charges and the potentials, while E_1', V_1, E_2', V_2, ... are another set of corresponding values. Hence if p_{11}', p_{12}', ... denote the values of the coefficients of induction for the displaced position of the system

$$V_1 = p_{11}'E_1' + p_{12}'E_2' + \ldots$$
$$V_2 = p_{12}'E_1' + p_{22}'E_2' + \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and

$$V_1' = p_{11}'E_1 + p_{12}'E_2 + \ldots$$
$$V_2' = p_{12}'E_1 + p_{22}'E_2 + \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Thus $$E_1V_1 + E_2V_2 + \ldots = p_{11}'E_1E_1' + p_{22}'E_2E_2' + \ldots + p_{12}'(E_1F_2' + E_1'E_2) + \ldots$$

and $$E_1'V_1' + E_2'V_2' + \ldots = p_{11}'E_1E_1' + p_{22}'E_2E_2' + \ldots + p_{12}'(E_1E_2' + E_1'E_2) + \ldots,$$

hence $$E_1V_1 + E_2V_2 + \ldots - (E_1'V_1' + \ldots) = 0.$$

Thus the difference between the loss in electric energy when the charges are kept constant and the gain when the potentials are kept constant is equal to

$$\tfrac{1}{2}\{(E_1 - E_1')(V_1 - V_1') + \ldots\}.$$

Now when the displacements are very small $E - E'$ and $V - V'$ will each be proportional to the first power of the displacements, and hence the preceding expression is proportional to the square of the displacements, and may be neglected when the displacements are very small. Hence we see that the loss in electric energy for any small displacement when the charges are kept constant, is equal to the gain in potential energy for the same displacement, when the potentials are kept constant. When the potentials are kept constant, the batteries which maintain the potentials of the conductors at their con-

stant value, will be called upon to furnish twice the amo mechanical work done by the electric forces. For they will furnish energy equal to the sum of the mechanical work done increase in the electric energy of the system; the latter is, as just seen, equal to the decrease in the electric energy of the while the charges are kept constant, and this is equal by the p of the Conservation of Energy to the mechanical work done.

37. Mechanical Force on each unit of area of a c conductor. The electric intensity is at right angles to the of the conductor, so that the force on any small portion of the surrounding a point P will be along the normal to the surfac

To find the magnitude of this force let us consider a small fied area round P. Then the electric intensity in the neighb of P may conveniently be regarded as arising from two (1) the electrification on the small area round P, and (2) the fication on the rest of the surface of the conductor and on a surfaces there may be in the electric field. To find the force small area we must find the value of the second part of the intensity, for the electric intensity due to the electrification small area will evidently not have any tendency to move t one way or another.

Let R be the total electric intensity along the outward normal just outside the surface at P, R_1 that part of it du electrification on the small area round P, R_2 the part due to the fication of the rest of the system. Then $R = R_1 + R_2$.

Compare now the electric intensities at two points Q, S near to P, but so placed that Q outside and S just inside the su which the small area forms a part tance between them being small c with the linear dimensions of t Then the part of the electric inte S in the direction of the outward at P, which is due to the electr on the conductors other than t area, will be equal to R_2 its value a

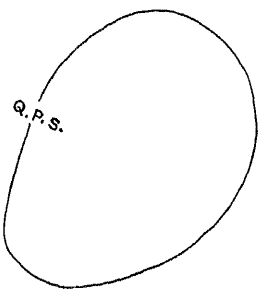


Fig. 15

these points are close together. The part of the electric inten

e small area will have at S the same magnitude as at Q, but will be
e opposite direction, since Q is on one side of the small area, while
on the other. Thus the electric intensity at S due to this area
e direction of the outward drawn normal will be $-R_1$, that due
e rest of the electrification R_2. The total intensity at S will
fore be $-R_1+R_2$. But this must be zero, since the intensity
e a closed equipotential surface enclosing no charge is zero. Thus
R_1, and therefore since

$$R = R_1 + R_2,$$

$$R_2 = \tfrac{1}{2}R.$$

ow the force on the area ω in the direction of the outward
al is $R_2\omega\sigma$ if σ is the surface density at P; thus if F is the
anical force per unit area in the direction of the outward
al

$$F\omega = R_2\omega\sigma = \tfrac{1}{2}R\omega\sigma,$$

$$F = \tfrac{1}{2}R\sigma \quad\text{.................................(1).}$$

ince by Coulomb's Law, Art. 22,

$$R = 4\pi\sigma,$$

ave the following expressions for the force per unit area

$$F = \frac{R^2}{8\pi} \quad\text{............................(2),}$$

$$F = 2\pi\sigma^2 \quad\text{........................(3).}$$

ince Coulomb's Law requires modification when the medium
unding the conductor is not air, the expressions (2) and (3) are
true for air: the equation (1) is always true whatever be the
ator surrounding the conductor.

Vhen the electric intensity at the surface of a conductor exceeds
tain value the air ceases to insulate and the electrification of the
uctor is discharged. The value of the electric intensity when the
rification begins to escape from the conductor, depends upon
eat number of circumstances, such as the pressure of the air
the proximity of other conductors. When the pressure of the air
out 760 mm. of mercury and the temperature about 15° C., the
est value of R is about 100, unless the conductor is within a

fraction of a millimetre of other conductors; hence the greatest value of F in dynes per square centimetre is

$$10^4/8\pi.$$

The pressure of the atmosphere is about 10^6 dynes per square centimetre, hence the greatest tension along the normal to an electrified surface in air is about $1/800\pi$ of the atmospheric pressure. That is, a pressure due to about ·3 of a millimetre of mercury would equal in magnitude the greatest tension on a conductor placed in air at ordinary pressure.

CHAPTER II

LINES OF FORCE

Expression of the properties of the Electric Field in of Faraday Tubes. The results we have hitherto obtained epend upon the fact that two charged bodies are attracted s or repelled from each other with a force varying inversely square of the distance between them; we have made no as- on as to how this force is produced, whether, for example, e to the action at a distance of the charged bodies upon each or to some action taking place in the medium between the

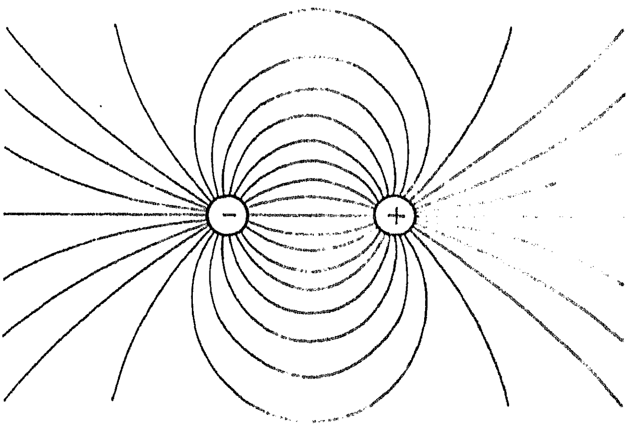


Fig. 16

at advances have been made in our knowledge of electricity the introduction by Faraday of the view that electrical are due to the medium between the charged bodies being in l state, and do not arise from any action at a distance exerted charged body on another.

shall now proceed to consider Faraday's method of regarding tric field—a method which enables us to form a vivid mental

picture of the processes going on in such a field, and to connect together with great ease many of the most important theorems in Electrostatics.

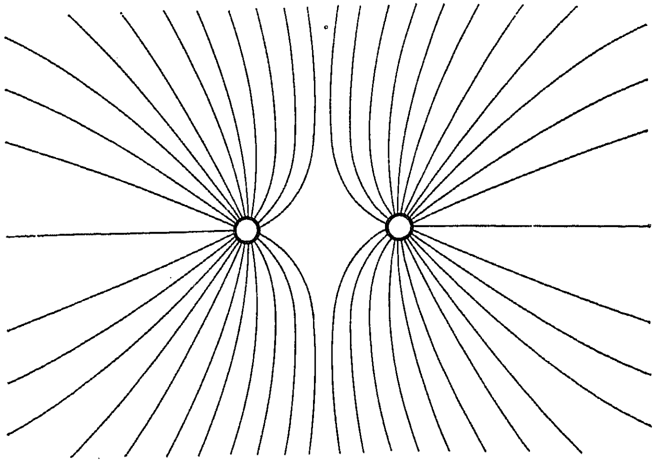


Fig. 17

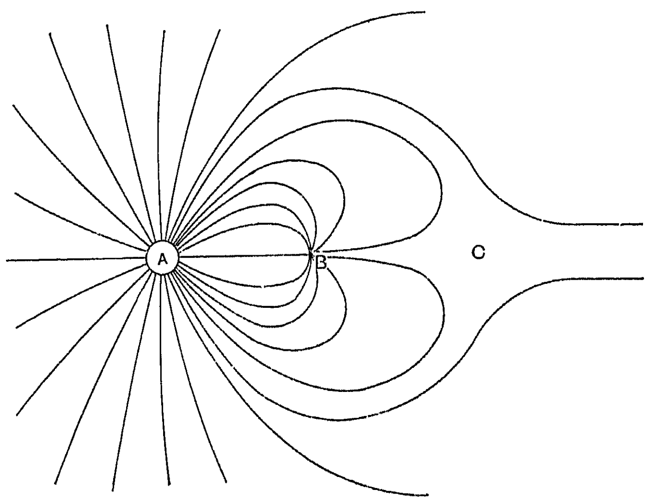


Fig. 18

We have seen in Art. 15 that a line of force is a curve such that its tangent at any point is in the direction of the electric intensity

at point. As these lines of force are fundamental in the method oyed in this and subsequent chapters for considering the pros of the electric field, we give below some carefully drawn ams of the lines of force in some typical cases.

gure 16 represents the lines of force due to two equal and oppoharges. In this case all the lines of force start from the positive e and end on the negative. Figure 17 represents the lines of force o two equal positive charges; in this case the lines of force do ass between the charged bodies, but lines start from each of the s and travel off to an infinite distance.

gure 18 represents the lines of force due to a positive charge to 4 at A, and a negative charge equal to -1 at B. In this case

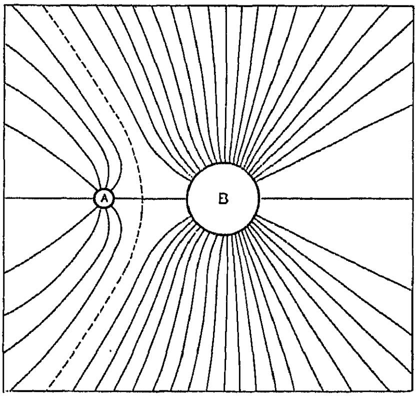


Fig. 19

lines of force which fall on B start from A, but since the charge s numerically greater than that at B, lines of force will start from ch do not fall on B but travel off to an infinite distance.

e lines of force which pass between A and B are separated from which proceed from A and go off to an infinite distance by the force which passes through C, the point of equilibrium, where

$$AC = 2AB.$$

gure 19 represents the lines of force due to a charge 1 at A and 3.

gure 20 represents the lines of force due to a charged conductor d by two spheres intersecting at right angles. The electric ity vanishes along the intersection of the spheres.

Figure 21 represents the lines of force between two finite parall places; between the plates but away from the edges of the plates th lines of force are straight lines at right angles to the planes, but nea

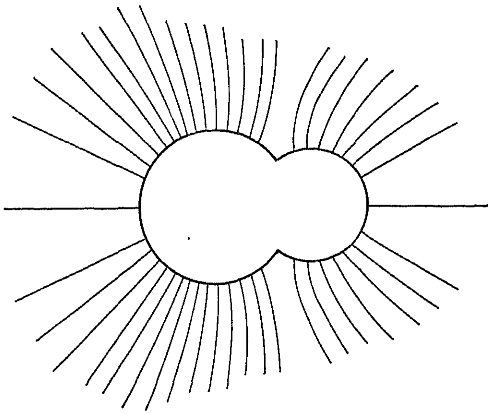


Fig. 20

the edges of the plates they curve out; some lines also pass from th back of one plate to the back of the other.

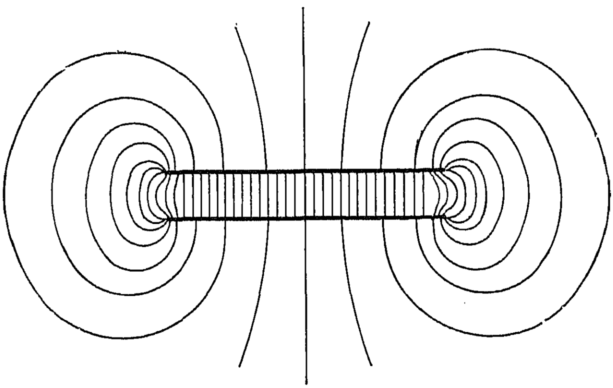


Fig. 21

39. Tubes of force. If we take any small closed curv in the electric field and draw the lines of force, which pass throug each point of the curve, these lines will form a tubular surface whic

ed a tube of force. These tubes possess the property that the
c intensities at any two points on a tube are inversely propor-
to the areas of the cross sections of the tube normal to the
f force at these points, provided that the cross sections are
all that the electric intensity may be regarded as constant
ach section. For let Fig. 22 represent a closed surface formed
tube and its normal sections.
be the area of the cross section
tube at P, ω_2 its cross section
R_1, R_2 the electric intensities
nd Q respectively. Now con-
he total normal electric induc-
er the surface. The only parts
surface which contribute any-
to this are the flat ends, as the
of the tube are by hypothesis
el to the electric intensity, so that this has no normal com-
t over the sides. Thus the total normal induction over the
surface PQ is equal to

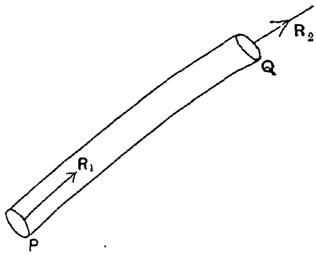


Fig. 22

$$R_2\omega_2 - R_1\omega_1;$$

nus sign being given to the second term because, as drawn in
ure, the electric intensity at P is in the direction of the inward-
normal. Now, by Gauss's theorem, the total normal electric
ion over any closed surface is equal to 4π times the charge
the surface; hence if the surface does not include any charge,
ve

$$R_2\omega_2 - R_1\omega_1 = 0,$$

electric intensity at P is to that at Q inversely as the cross
n of the tube of force at P is to that at Q.
e tubes of force will start from positive electrification and go on
hey end on a negative electrified body. If the points P and Q
the surfaces of positively and negatively electrified conductors,
f σ_P is the surface density at P, σ_Q that at Q,

$$R_1 = 4\pi\sigma_P, \quad R_2 = -4\pi\sigma_Q;$$

he equation $$R_2\omega_2 - R_1\omega_1 = 0,$$

valent to $$\sigma_Q\omega_2 = -\sigma_P\omega_1.$$

ce $\sigma\omega'$ is the charge of electricity on B, it is equal to N, the
r of Faraday tubes which start from B and which pass through
ce

$$F\omega = 4\pi N,$$

is unity

$$F = 4\pi N.$$

is the electric intensity at any point in air is 4π times the
r of Faraday tubes passing through unit area of a plane drawn
n the point at right angles to the electric intensity.

The properties of the Faraday tubes enable us to prove with
any important theorems relating to the electric field.
is, for example, we see that on the conductor at the highest
al in the field the electrification must be entirely positive.
gative electrification would imply that Faraday tubes arrived
conductor; these tubes must however arrive at a place which
lower potential than the place from which they start. Thus,
potential of the conductor we are considering is the highest
ield it is impossible for a Faraday tube to arrive at it, for this
imply that there was some other conductor at a still higher
al from which the tube could start.
ilar reasoning shows that the electrification on the conductor
ductors at the lowest potential in the field must be entirely
e.
en one conductor has a positive charge while all the other
tors are connected to earth, we see from the last result that
rges on the uninsulated conductors must be all negative, and
he potentials of these conductors are all equal and the same
of the earth, no Faraday tubes can pass from one of these
tors to another, or from one of these to the earth. Hence all
es which fall on these conductors must have started from the
tor at highest potential. Thus the sum of the number of tubes
fall on the uninsulated conductors cannot exceed the number
eave the positively charged conductor, that is, the sum of the
e charges induced on the conductors connected to earth cannot
the positive charge on the insulated conductor.

These results give us important information as to the co-
ts of capacity and of induction defined in Art. 26.
let us take the first conductor as the insulated one with the

positive charge; then since V_2, V_3,... are all zero we have, using th notation of that Article,

$$E_1 = q_{11}V_1, \quad E_2 = q_{12}V_1, \quad E_3 = q_{13}V_1, \ldots.$$

Since E_1 and V_1 are positive, while E_2, E_3, etc. are all negative we see that q_{11} is positive, while q_{12}, q_{13}, etc. are all negative. Again since the positive charge on the first conductor is numerically no less than the sum of the negative charges on the other conductors

$$E_1 \text{ is numerically not less than } E_2 + E_3 + \ldots,$$

i.e. q_{11} is numerically not less than $q_{12} + q_{13} + q_{14} + \ldots.$

If one of the conductors, say the second, completely surrounds th first, and if there is no conductor other than the first inside the second and if all the conductors except the first are at zero potential, then all the tubes which start from the first must fall on the second. Thu the negative charge on the second must be numerically equal to th positive charge on the first (see Art. 30). There can be no charge on any of the other conductors, for all the tubes which might fal on these conductors must come from the first conductor, and all th tubes from this conductor are completely intercepted by the secon surface. Thus if the second conductor encloses the first conductor and if there are no other conductors between the first and the second then $q_{11} = -q_{12}$, and $q_{13}, q_{14}, q_{15}, \ldots$ are all zero.

43. Expression for the Energy in the Field. When w regard the Faraday tubes as the agents by which the phenomena i the electric field are produced we are naturally led to suppose tha the energy in the electric field is in that part of the field through whic the tubes pass, i.e. in the dielectric between the conductors. We sha now proceed to find how much energy there must be in each unit o volume if we regard the energy as distributed throughout the electri field. We have seen in Art. 23, that the electric energy is one hal the sum of the products got by multiplying the charge on each con ductor by the potential of that conductor. We may regard each uni charge as having associated with it a Faraday tube, which commence at the charge if that is positive and ends there if the charge i negative. Let us now see how the energy in the field can be expresse in terms of these tubes. Each tube will contribute twice to the ex pression for the electric energy $\frac{1}{2}\Sigma EV$, the first time corresponding t

the positive charge at its origin, the second time corresponding to the negative charge at its end. Thus, since there is unit charge at each end of the tube, the contribution of each tube to the expression for the energy will be $\frac{1}{2}$ (the difference of potential between its beginning and end). The difference of potential between the beginning and end of the tube is equal to $\Sigma(R \, . \, PQ)$, where PQ is a small portion of the length of the tube so small that along it R, the electric intensity, may be regarded as constant; the sign Σ denotes that the tube between A and B, A being a unit of positive and B a unit of negative charge, is to be divided up into small pieces similar to PQ, and that the sum of the products of the length of each piece into the electric intensity along it is to be taken. Thus the whole tube AB contributes $\frac{1}{2}\Sigma R \, . \, PQ$ to the electric energy, so that we may suppose that each unit length of the tube contributes an amount of energy equal to one half the electric intensity. Any finite portion CD of the tube will therefore contribute an amount of energy numerically equal to one half the difference of potential between C and D. We may therefore regard the electrical energy as distributed throughout the field and that each of the Faraday tubes has associated with it an amount of energy per unit length numerically equal to one half the electric intensity.

Let us now consider the amount of energy per unit volume. Take a small cylinder surrounding any point P in the field with its axis parallel to the electric intensity at P, its ends being at right angles to the axis. Then if R is the electric intensity at P and l the length of the cylinder, the amount of energy due to each tube passing through the cylinder is $\frac{1}{2}Rl$. If ω is the area of the cross section of the cylinder, N the number of tubes passing through unit area, the number of tubes passing through the cylinder is $N\omega$. Thus the energy in the cylinder is

$$\tfrac{1}{2}RlN\omega,$$

but in air, by Art. 40,

$$4\pi N = R,$$

so that the energy in the cylinder is

$$\frac{1}{8\pi}R^2l\omega.$$

But $l\omega$ is the volume of the cylinder, hence the energy per unit volume is equal to

$$\frac{R^2}{8\pi}.$$

Thus we may regard the energy as distributed throughout th field in such a way that the energy per unit of volume is equal t

$$R^2/8\pi.$$

44. If we divide the field up by a series of equipotential surfaces the potentials of successive surfaces decreasing in arithmetical pro gression, and if we then draw a series o tubular surfaces cutting these equipoten tial surfaces at right angles, such that th number of Faraday tubes passing throug the cross section of each of the tubula surfaces is the same for all the tubes, th electric field will be divided up into number of cells which will all contain th same amount of energy. For the potentia difference between the places where Faraday tube enters and leaves a cell i the same for all the cells, and thus the energy of the portion of eac Faraday tube passing through a cell will be constant for all the cells and since the same number of Faraday tubes pass through eac cell, the energy in each cell will be constant.

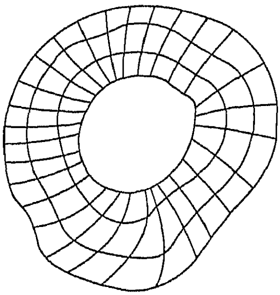


Fig. 23

45. Force on a conductor regarded as arising from th Faraday Tubes being in a state of tension. We hav seen, Art. 37, that on each unit of area of a charged conductor ther is a pull equal to $\frac{1}{2}R\sigma$, where σ is the surface density of the electricit and R the electric intensity. Now σ is equal to the number of Farada tubes which fall on unit area of the surface, and hence the force on th surface is the same as if each of the tubes exerted a pull equal to $\frac{1}{2}R$ Thus the mechanical forces on the conductors in the electric fiel are the same as they would be if the Faraday tubes were in a stat of tension, the tension at any point being equal to one half the electri intensity at that point. Thus the tension at any point of a Farada tube is numerically equal to the energy per unit length of the tub at that point.

If we have a small area ω, at right angles to the electric intensity the tension over this area is equal to

$$\tfrac{1}{2}NR\omega,$$

where N is the number of Faraday tubes passing through unit area, and R is the electric intensity. By Art. 40

$$N = \frac{R}{4\pi}.$$

Hence the tension parallel to the electric intensity is

$$\frac{1}{8\pi} R^2 \omega.$$

The tension across unit area is therefore equal to

$$\frac{R^2}{8\pi}.$$

46. This state of tension will not however leave the dielectric in equilibrium unless the electric field is uniform, that is unless the tubes are straight and parallel to each other. If however there is in

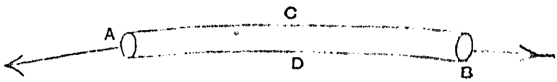


Fig. 24

addition to this tension along the lines of force a pressure acting at right angles to them and equal to $R^2/8\pi$ per unit area the dielectric will be in equilibrium, and since this pressure is at right angles to the electric intensity it will not affect the normal force acting on a conductor. To show that this pressure is in equilibrium with the tensions along the Faraday tubes, consider a small volume whose ends are portions of equipotential surfaces and whose sides are lines of force.

Let us now consider the forces acting on this small volume parallel to the electric intensity at A (Fig. 24). The forces are the tensions in the Faraday tubes and the pressures at right angles to the sides. Resolve these parallel to the outward-drawn normal at A. The number n' of Faraday tubes which pass through A is the same as the number which pass through B. If R, R' are the electric intensities at A and B respectively, then the force exerted on the volume in the direction of the outward-drawn normal at A by the Faraday tubes at A will be $n'R/2$, while the force in the opposite direction exerted by the Faraday tubes at B is $n'R' \cos \epsilon/2$, where ϵ is the small angle between the

directions of the Faraday tubes at A and B. Since ϵ is a very small angle we may replace $\cos \epsilon$ by unity; thus the resultant force on the volume in the direction of the outward-drawn normal at A due to the tension in the Faraday tubes is

$$n'(R - R')/2.$$

Let N be the number of tubes passing through unit area, ω, ω' the areas of the ends A and B respectively; then, Art. 40,

$$n' = N\omega = \frac{R}{4\pi}\omega = \frac{R'}{4\pi}\omega',$$

so that the resultant in the direction of the outward-drawn normal at A is

$$\frac{R}{4\pi}\omega(R - R')/2;$$

since $$R'\omega' = R\omega,$$

we may write this as $$\frac{RR'}{8\pi}(\omega' - \omega),$$

or approximately, since R' is very nearly equal to R

$$\frac{R^2}{8\pi}(\omega' - \omega).$$

Let us now consider the effect of the pressure p at right angles to the lines of force; this has a component in the direction of the outward-drawn normal at A as in consequence of the curvature of the tube the normals to its surface are not everywhere at right angles to this direction; the angle between the pressure and the normal at A will always however be nearly a right angle. If this angle is $\frac{\pi}{2} - \theta$ at a point where the pressure is p', the component of the pressure along the normal at A will be proportional to $p' \sin \theta$. But since p' only differs from p, the value of the pressure at A, by a small quantity, and θ is small, the component of the pressure will, if we neglect the squares of small quantities, be equal to $p \sin \theta$; that is, the effect along the normal at A of the pressure over the surface will be approximately the same as if that pressure were uniform. To find the effect of the pressure over the sides we remember that a uniform hydrostatic pressure over any closed surface is in equilibrium; hence the force due to the pressures over the sides C, D will be equal and

opposite to the force due to equal pressures over the ends A and B. But the force due to the pressure over these ends is $p\omega' - p\omega$; hence the resultant effect in the direction of the outward-drawn normal at A of the pressure over the sides is $p(\omega - \omega')$. Combining this with the effect due to the tension in the tubes we see that the total force on the element parallel to the outward-drawn normal at A is

$$\frac{R^2}{8\pi}(\omega' - \omega) + p(\omega - \omega');$$

this vanishes if $$p = \frac{R^2}{8\pi} = \frac{NR}{2}.$$

Thus the introduction of this pressure will maintain equilibrium as far as the forces parallel to the electric intensity are concerned.

Now consider the force at right angles to the electric intensity. Let $PQRS$, Fig. 25, be the section of the volume AB in Fig. 24 by the

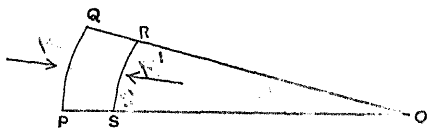


Fig. 25

plane of the paper, PS, QR being sections of equipotential surfaces, and PQ, SR lines of force. Let t be the depth of the volume at right angles to the plane of the paper. We shall assume that the section of the figure by the plane through PQ at right angles to the plane of the paper is a rectangle. Let R be the electric intensity along PQ, R' that along SR, s the length PQ, s' that of SR. Since the difference of potential between P and Q is the same as that between S and R,

$$Rs = R's'.$$

Consider the forces parallel to PS. First take the tensions along the Faraday tubes; the force due to those at PS will have no component along PS: in each tube at QR there is a tension $R/2$, the component of which along PS is $(R \sin \theta)/2$, where θ is the angle between PS and QR. Since θ is very small this component is equal to $R\theta/2$. Let PS and QR meet in O,

$$\theta = \frac{RS}{OR} = \frac{PQ}{OQ} = \frac{PQ - SR}{OQ - OR} = \frac{s - s'}{RQ}.$$

Thus the component along PS due to the tension at QR is

$$\frac{R}{2} \cdot \frac{s - s'}{RQ}.$$

The number of tubes which pass through the end of the volume at RQ is $N \,.\, QR \,.\, t$, where N is the number of tubes which pass through unit area.

The total component along PS due to the tensions in these tubes is thus

$$\frac{R}{2} \frac{(s - s')}{RQ} \,.\, N \,.\, QR \,.\, t$$
$$= \frac{R^2}{8\pi} (s - s')\, t.$$

Consider now the pressures at right angles to the lines of force. The component along PS due to these pressures is equal to

$$pst - p's't,$$

where p and p' are the pressures over PQ, RS respectively.

If
$$p = \frac{R^2}{8\pi}, \quad p' = \frac{R'^2}{8\pi};$$
$$pst - p's't = \left(\frac{R^2}{8\pi} s - \frac{R'^2}{8\pi} s'\right) t$$
$$= \frac{RR'}{8\pi} (s' - s)\, t, \text{ (since } Rs = R's'),$$

or approximately, since R' is very nearly equal to R,

$$= \frac{R^2}{8\pi} (s' - s)\, t.$$

Thus the component in the direction of PS due to the tensions is equal and opposite to the component due to the pressures; thus the two are in equilibrium as far as the component in the plane of the diagram at right angles to the electric intensity is concerned; we easily see that the same is true for the component at right angles to the plane of the paper. We have already proved that the tensions and pressures balance as far as the component along the direction of the electric intensity is concerned; thus the system of pressures and tensions constitutes a system in equilibrium.

47. This system of tensions along the tubes of force and pressures at right angles to them is thus in equilibrium at any part of the dielectric where there is no charge, and gives rise to the forces which act on electrified bodies when placed in the electric field. Faraday introduced this method of regarding the forces in the electric field; he expressed the system of tensions and pressures which we have just found, by saying that the tubes tended to contract and that they repelled each other. This conception enabled him to follow the processes of the electric field without the aid of mathematical analysis.

Since $$Rs = R's',$$

and $$\frac{s}{s'} = \frac{OQ}{OR} = 1 + \frac{RQ}{OR},$$

we have $$\frac{R - R'}{RQ} = -\frac{R}{OR}.$$

Now OR is the radius of curvature of the line of force; denoting this by ρ we have

$$\frac{1}{\rho} = -\frac{\frac{dR}{d\nu}}{R},$$

where $d\nu$ is an element of length at right angles to the electric force; we see from this equation that the lines of force are concave to the stronger parts of the field.

The lines of force arrange themselves as a system of elastic strings would do if acted on by forces whose potential for unit length of string was $R/2$.

48. The student will find that much light is thrown on the effects produced in the electric field by the careful study from this point of view of the diagrams of the lines of force given in Art. 38. Thus, take as an example the diagram given in Fig. 18, which represents the lines of force due to two charges A and B of opposite signs, the ratio of the charges being 4 : 1. We see from the diagram that though more tubes of force start from the larger charge A, and the tension in each of these is greater than in a tube near the smaller charge B, the tubes are much more symmetrically distributed round A than round B. The approximately symmetrical distribution of the tubes round A makes the pulls exerted on A by the taut Faraday tubes so nearly counterbalance each other that the resultant pull of these tubes on A

is only the same as that exerted on B by the tubes starting from it; since these, though few in number, are less symmetrically distributed, and so do not tend to counterbalance each other to nearly the same extent. The tubes of force in the neighbourhood of the point of equilibrium are especially interesting. Since the charge on A is four times that on B, only $\frac{1}{4}$ of the tubes which start from A can end on B, the remaining $\frac{3}{4}$ must go off to other bodies, which in the case given in the diagram are supposed to be at an infinite distance. The point of equilibrium corresponds as it were to the 'parting of the ways' between the tubes of force which go from A to B and those which go off from A to an infinite distance.

When the charges A and B are of the same sign, as in Fig. 19, we see how the repulsion between similar tubes causes the tubes to congregate on the side of A remote from B, and on the side of B remote from A.

We see again how much more symmetrically the tubes are distributed round A than round B; this more symmetrical distribution of the tubes round A makes the total pull on A the same as that on B.

We see too from this example that the repulsion between the charges of the same sign and the attraction between charges of opposite signs are both produced by the same mechanism, i.e. a system of pulls; the difference between the cases being that the pulls are so distributed that when the charges are of the same sign the pulls tend to pull the bodies apart, while when the bodies are of opposite signs the pulls tend to pull the bodies together.

The diagram of the lines of force for the two finite plates (Fig. 21) shows how the Faraday tubes near the edges of the plates get pushed out from the strong parts of the field and are bent in consequence of the repulsion exerted on each other by the Faraday tubes.

49. As an additional example of the interpretation of the processes in the electric field in terms of the Faraday tubes, let us consider the effect of introducing an insulated conductor into an electric field.

Let us take the field due to a single positively charged body at A; before the introduction of the conductor the Faraday tubes were radial, but when the conductor is introduced the tubes, which pre-

existed in the region occupied by the conductor, are annulled; e repulsion previously exerted by these tubes on the surg ones ceases, and a tube such as AB, which was previously, is now, since the pressure below it is diminished, bent down the conductor; the tubes passing near the conductor may down so much that they strike against it, they then divide ı two tubes, with negative electrification at the end C, positive nd D.

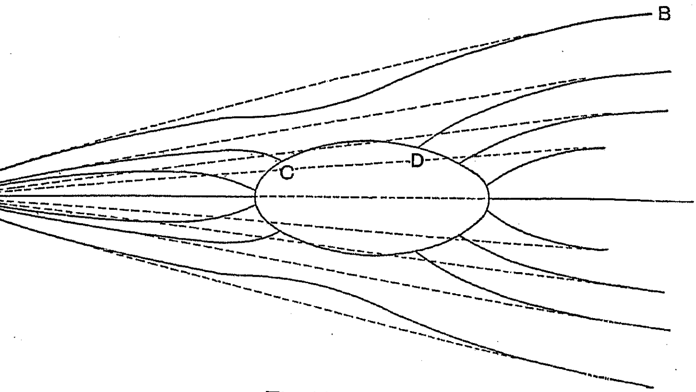


Fig. 26

Force on an uncharged conductor placed in an field. If a small conductor is placed in the field at P, day tubes inside the conductor disappear, and, if the intro-of the conductor did not alter the tubes outside it, the dimif energy due to the annihilation of the tubes in the conductor e proportional to $R^2/8\pi$ per unit volume, where R is the ntensity in the field at P before the conductor was intro-If the conductor is moved to a place where the electric ins R', the diminution in the electric energy in the field is er unit volume. Now it is a general principle in mechanics ystem always tends to move from rest in such a way as to the potential energy as much as possible, and the force to assist a displacement in any direction is equal to the liminution of the potential energy in that direction. The r will thus tend to move so as to produce the greatest posinution in the electric energy, that is, it will tend to get

into the parts of the field where the electric intensity is as large as possible; it will thus move from the weak to the strong parts of the field.

The presence of the conductor will however disturb the electric field in its neighbourhood; thus **R**, the actual electric intensity, will differ from R, the electric intensity at the same point before the conductor was introduced. By differentiating $R^2/8\pi$ we shall get an inferior limit to the force acting on the conductor per unit volume. For suppose we introduce a conductor into the electric field, then $R^2/8\pi$ would be the diminution in electric energy per unit volume due to the disappearance of the Faraday tubes from the inside of the conductor, the tubes outside being supposed to retain their original position. In reality however the tubes outside will have to adjust themselves so as to be normal to the conductor, and this adjustment will involve a further diminution in the energy, thus the actual change in the energy is greater than that in $R^2/8\pi$ and the force acting per unit volume will therefore be greater than the rate of diminution of this quantity. If we take the case when the force is due to a charge e at a point, the rate of diminution of $R^2/8\pi$ is $e^2/2\pi r^5$, and thus the force on a small conducting sphere of radius a will be greater than $(4\pi a^3/3)(e^2/2\pi r^5)$, that is, greater than $2e^2a^3/3r^5$. The actual value (see Art. 87) is $2e^2a^3/r^5$.

CHAPTER III

CAPACITY OF CONDUCTORS. CONDENSERS

The capacity of a conductor is defined to be the numerical f the charge on the conductor when its potential is unity, ther conductors in the field being at zero potential.

conductors insulated from each other and placed near to-orm what is called a condenser; in this case the charge on onductor may be large, though the difference between their ls is small.

any instruments the two conductors are so arranged that arges are equal in magnitude and opposite in sign; in such e magnitude of the charge on either conductor when the l difference between the conductors is unity is called the of the *condenser*.

e difference of potential between two conductors, produced g a charge $+q$ to one conductor and $-q$ to the other, is V, V is defined to be the capacity between the conductors.

Capacity of a Sphere placed at an infinite distance ther conductors. Let a be the radius of the sphere, otential, e its charge, the corresponding charge of opposite ng at an infinite distance. Then (Art. 17), the potential due harge on the sphere at a distance r from the centre is e/r; e the potential at the surface of the sphere is e/a.

ce we have

$$V = \frac{e}{a}.$$

n V is unity, e is numerically equal to a; hence, Art. 51, the of the sphere is numerically equal to its radius.

Capacity of two concentric spheres. Let us first case when the outer sphere and any conductors which may de it are connected to earth, while the inner sphere is main-t potential V. Then, since the outer sphere and all the con-

ductors outside are connected to earth, no Faraday tubes can start from or arrive at the outer surface of the outer sphere, for Faraday tubes only pass between places at different potentials, and the potentials of all places outside the sphere are the same, being all zero. Again, all tubes which start from the inner sphere will arrive at the internal surface of the outer shell, so that the charge on the inner surface of this shell will be equal and opposite to the charge on the inner sphere. Let a be the radius of the inner sphere, b the radius of the internal surface of the outer sphere, e the charge on the inner sphere, then $-e$ will be the charge on the interior of the outer sphere.

Consider the work done in moving a unit of electricity from the surface of the inner sphere to the inner surface of the outer sphere; the charge on the outer sphere produces no electric intensity at a point inside, so that the electric intensity, which produces the work done on the unit of electricity, arises entirely from the charge on the inner sphere. The electric intensity due to the charge on this sphere is, by Art. 11, the same as that which would be due to the charge e collected at the centre O. The work done on unit of electricity when it moves from the inner sphere to the outer one is thus the same as the work done on a unit charge when it moves from a distance a to a distance b from a small charged body placed at the centre of the spheres; this, by Art. 17, is equal to

$$\frac{e}{a} - \frac{e}{b},$$

and is by definition equal to V, the potential difference between the two spheres; hence we have

$$V = \frac{e}{a} - \frac{e}{b},$$

or

$$e = \frac{ab}{b-a} V.$$

Thus, when $b - a$ is very small, that is, when the radii of the two spheres are very nearly equal, the charge is very large. When $V = 1$, the charge is

$$\frac{ab}{b-a};$$

so that this is, by Art. 51, the capacity of the two spheres. The value of this quantity when the radii of the two spheres are very nearly

qual is worthy of notice. In this case, writing t for $b - a$, the dis-
ance between the spheres, the capacity is equal to

$$\frac{ab}{t} = \frac{a(a+t)}{t};$$

his, since t is very small compared with a, is approximately

$$\frac{a^2}{t} = \frac{4\pi a^2}{4\pi t}$$

$$= \frac{\text{surface of the sphere}}{4\pi t}.$$

hus the capacity in this case is equal per unit area of surface to $1/4\pi$ mes the distance between the conductors. The case of two spheres hose distance apart is very small compared with their radii is how- ver approximately the case of two parallel planes; hence the capa- ity of such planes per unit area of surface is equal to $1/4\pi$ times the istance between the planes. This is proved directly in Art. 56.

If, after the spheres are charged, the inner one is insulated, and he outer one removed to an infinite distance (to enable this to be one we may suppose that the outer sphere consists of two hemi- pheres fitted together, and that these are separated and removed), he charge on the sphere will remain equal to e, i.e. $\frac{ab}{b-a}V$, but he potential of the sphere will rise; when it is alone in the field the otential will be e/a, i.e.

$$\frac{b}{b-a}V.$$

Thus by removing the outer sphere the potential difference etween the sphere and the earth has been increased in the propor- ion of b to $b - a$. By making $b - a$ very small compared with b, ve can in this way increase the potential difference enormously and nake it capable of detection by means which would not have been ufficiently sensitive before the increase in the potential took place.

It was by the use of this principle that Volta succeeded in demon- trating by means of the gold-leaf electroscope and two metal plates, he difference of potential between the terminals of a galvanic cell; his difference is so small that the electroscope is not deflected when he cell is directly connected to it; by connecting the terminals of the ell to two plates placed very close together, and then removing one

of the plates *after* severing the connections between the plates a the cells, Volta was able to increase the potential of the other pl to such an extent that it produced an appreciable deflection of electroscope with which it was connected.

Work has to be done in separating the two conductors; this w appears as increased electric energy. Thus, to take the case of two spheres, when both spheres were in position the electric ener which, by Art. 23, is equal to $\frac{1}{2}\Sigma EV$, is

$$\frac{1}{2}\frac{ab}{b-a}V^2.$$

When the outer sphere which is at zero potential is removed potential of the sphere is e/a, so that the electric energy is

$$\frac{1}{2}\frac{e^2}{a},$$

or $$\frac{1}{2}\frac{ab^2}{(b-a)^2}V^2,$$

and has thus been increased in the proportion of b to $b-a$.

54. Let us now take the case when the inner sphere is connec to earth while the outer sphere is at the potential V. In this ca we can prove exactly as before that the charge on the inner sph is equal and opposite to the charge on the internal surface of outer sphere, and that, if e is the charge on the inner sphere,

$$e = \frac{ab}{b-a}V.$$

In this case, in addition to the positive charge on the internal surf of the outer sphere, there will be a positive charge on the exter surface, since this surface is at a higher potential than the surround conductors. If c is the radius of the external surface of the ou sphere, the sum of the charges on the two spheres must be Vc. Si the charge on the inner surface of the outer sphere is equal and op site to the charge on the inner sphere, the charge on the exter surface of the outer sphere must be equal to Vc. Thus the to charge on the outer sphere is equal to

$$\frac{ab}{b-a}V + cV.$$

The charge on the outside of the outer sphere will be affected presence of other conductors. Let us suppose that outside ernal sphere there is a small sphere connected to earth; the radius of this sphere, R the distance of its centre from O re of the concentric spheres. Let e' be the total charge on concentric spheres, e'' the charge on the small sphere. The l due to e' at a great distance R from O is e'/R, similarly the l due to e'' is at a distance R equal to e''/R.

e the surface of the outer sphere is at the potential V, we

$$V = \frac{e'}{c} + \frac{e''}{R},$$

ce the potential of the small sphere is zero, we have

$$0 = \frac{e'}{R} + \frac{e''}{r},$$

$$V = \frac{e'}{c}\left\{1 - \frac{rc}{R^2}\right\},$$

$$e' = \frac{cV}{1 - \frac{rc}{R^2}},$$

the presence of the small sphere increases the charge on the here in the proportion of

$$1 \text{ to } 1 - rc/R^2.$$

only the charge on the external surface of the outer sphere affected. The charges on the inner sphere and on the internal of the outer sphere are not altered by the presence of con- outside the latter sphere.

Parallel Plate Condensers. Condensers are frequently cted of two parallel metallic plates; the theory of the case, e plates are so large in comparison with their distance apart y may be regarded as infinite in area, is very simple.

his case the Faraday tubes passing between the plates will ght and at right angles to the plates, and the electric intensity the plates is constant since in passing from one plate to the ach Faraday tube has a constant cross section; let R be its hen if d is the distance between the plates, the work done

on unit charge of electricity as it passes from the plate where the potential is high to the one where the potential is low is Rd, and this by definition is equal to V, the difference of potential between the plates. Hence

$$V = Rd.$$

If σ is the surface-density of the charge on the plate at high potential, that on the plate of low potential will be $-\sigma$, and by Coulomb's law, Art. 22,

$$R = 4\pi\sigma.$$

Hence
$$V = 4\pi\sigma d,$$

or
$$\sigma = \frac{V}{4\pi d} \quad\text{.............................(1)},$$

and if V is equal to unity, σ is equal to

$$\frac{1}{4\pi d}.$$

The charge on an area A of one of the plates when the potential difference is unity is thus $A/4\pi d$, this by definition is the capacity of the area A. We arrived at the same result in Art. 53 from the consideration of two concentric spheres. The electrical energy of the condenser is, by Art. 23, equal to

$$\tfrac{1}{2}\Sigma EV,$$

which in this case is equal to

$$\frac{V^2 A}{8\pi d},$$

or, if E is the charge on one of the plates, to

$$\frac{2\pi d E^2}{A}.$$

57. Guard Ring. In practice it is of course impossible to have infinite plates, and when the plates are finite, then, as the diagram, Fig. 21, Art. 38, shows, the Faraday tubes near the edges of the plates are no longer straight, and the electrification ceases to be uniform, and is no longer given by the expression (1), Art. 56. Thus to express the quantity of electricity on the finite plate, we should have to add to the expression a correction for the inequality of the distribution over the ends of the plates. This correction can be calculated, but the necessity for it may be avoided in practice by making use of a device due to Lord Kelvin, and called a guard ring.

ppose one of the plates, say the upper one, is divided into three ns flush with each other and separated by the narrow gaps Then if, in charging the condenser, the portions A, B, C are cted metallically with each other, the places where the electrifi- is not uniform will be on A and C, so that apart from the of the narrow gaps E, F, the electrification on B will, if we t the effect of the gaps, be uniform and the total charge on B e equal to $SV/4\pi d$, where S is the area of the plate B. The ty of B is thus equal to $S/4\pi d$.

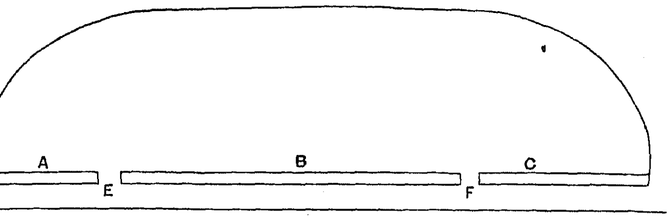


Fig. 27

as ought to be the case, the widths of the gaps at E and F are mall compared with the distance between the plates, we can calculate the effect of the gaps. For if the gaps are very narrow ectrification of the lower plate will be approximately uniform. araday tubes in the neighbourhood of the gaps will be distri- as in Fig. 28. We see from this, if we consider the gap E, that e Faraday tubes which would have fallen on a plate whose h was E, if there had been no gap, will fall on one or other of ates A and B, Fig. 28, and from the symmetry of the arrange- half of these tubes will fall on e other half on A; thus the amount of electricity on B the same as if we supposed B end halfway across the gap,

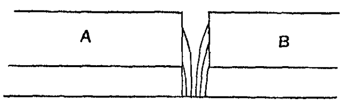


Fig. 28

be uniformly charged with electricity whose surface density πd. We see then that, allowing for the effects of the gaps, the ty of B will be equal to $S'/4\pi d$, where

S' = area of plate B
+ $\frac{1}{2}$ (the sum of the areas of the gaps E and F).

If the plate B is not at zero potential, there will be some electri cation on the back of the plate arising from Faraday tubes which g from the back of B to other conductors in its neighbourhood and earth. The electrification of the back of B may be obviated by cove ing this side of A, B, C with a metal cover connected with A and It can also be obviated by making B the low potential plate (i.e. t one connected to earth), care being taken that the other conducto in the neighbourhood are also connected to earth.

58. Capacity of two coaxial cylinders. Let us take t case of two coaxial cylinders, the inner one being at potential V, a the outer one at potential zero. Then if E is the charge per un length on the inner cylinder, $-E$ will be the charge per unit leng on the inner surface of the outer one, since all the Faraday tub which start from the inner cylinder end on the outer one.

The electric intensity at a distance r from the axis of the cylind is, by Art. 13, equal to

$$\frac{2E}{r}.$$

Thus the work done on unit charge, when it goes from the out surface of the inner cylinder to the inner surface of the outer cylind is equal to

$$\int_a^b \frac{2E}{r}\,dr,$$

where a is the radius of the inner cylinder, b the radius of the inn surface of the outer cylinder.

This work is, however, by definition equal to V, the differen of potential between the cylinders, and hence

$$V = \int_a^b \frac{2E}{r}\,dr$$

$$= 2E\log\frac{b}{a}.$$

When V is unity, E, the charge per unit length, is equal to

$$\frac{1}{2\log\frac{b}{a}},$$

and this, by definition, is the capacity of the condenser per u length.

If the radii of the cylinders are nearly equal, and if $b - a = t$,
ll be small compared with a; in this case the capacity per unit
gth

$$= \frac{1}{2 \log \frac{a+t}{a}}$$

$$= \frac{1}{2\frac{t}{a}} \text{ approximately}$$

$$= \frac{1}{2}\frac{a}{t}$$

$$= \frac{2\pi a}{4\pi t}.$$

Since $2\pi a$ is the area of unit length of the inner cylinder, the
acity per unit area is $1/4\pi t$; we might have deduced this result
n the case of two parallel planes.

When the two cylinders are coaxial, there is no force tending to
ve the inner cylinder; thus since the system is in equilibrium, the
ential energy, if the charges are given, must be either a maximum
a minimum. The equilibrium is, however, evidently unstable, for,
he inner cylinder is displaced, the force due to the electric field
ds to make the cylinders come into contact with each other and
s increase the displacement. Since the equilibrium is unstable the
ential energy is a maximum when the cylinders are coaxial. The
ential energy, however, is, by Art. 23, equal to

$$\frac{1}{2}EV = \frac{1}{2}\frac{E^2}{C},$$

ere C is the capacity of the condenser. Thus if the potential energy
a maximum the capacity must be a minimum. Thus any displace-
nt of the inner cylinder will produce an increase in the capacity,
l since the capacity is a minimum when the cylinders are coaxial,
increase in the capacity will be proportional to squares and higher
wers of the distance between the axes of the cylinders.

59. Condensers whose capacities can be varied. For
ne experimental purposes it is convenient to use a condenser whose
acity can be altered continuously, and in such a way that the
eration in the capacity can be easily measured. For this purpose

a condenser made of two parallel plates, one of which is fixed, while the other can be moved by means of a screw, through known distances, always remaining parallel to the fixed plate, is useful. In this case the capacity is inversely proportional to the distance between the plates, provided that this distance is never greater than a small fraction of the radius of the plates.

Another arrangement which has been used for this purpose is shown in Fig. 29. It consists of three coaxial cylinders, two of which, AB, CD, are of the same radius and are insulated from each other, while the third, EF, is of smaller radius and can slide parallel to its axis. The cylinder EF is connected metallically with CD, so that these two are always at the same potential, and the cylinder AB is at a different potential, then when the cylinder EF is moved about so as to expose different amounts of surface to AB the capacity of the condenser formed by AB and EF will alter, and the increase in the capacity will be proportional to the increase in the area of the surface of EF brought within AB.

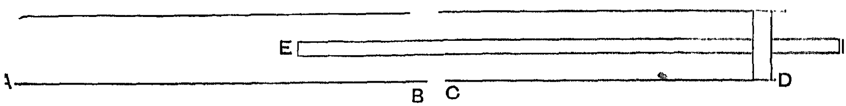


Fig. 29

60. Electrometers.

Consider the case of two parallel conducting plates; let V be the potential difference between the plates, d their distance apart. The force on a conductor per unit area is, by Art. 37, equal to $\frac{1}{2}R\sigma$, where R is the electric intensity at the conductor and σ the surface density; but $R = \frac{V}{d}$, while $\sigma = \frac{1}{4\pi}R$ by Coulomb's law; we see therefore that the attraction of one plate on the other is per unit area equal to

$$\frac{1}{8\pi}\frac{V^2}{d^2}.$$

Hence the force on an area A of one of the plates is equal to

$$\frac{A}{8\pi}\frac{V^2}{d^2} \quad \text{.............................} (1).$$

Thus, if we measure the mechanical force between the plates, we can deduce the value of V, the potential difference between them.

rounded by a fixed guard ring in a definite position, when this known the value of the potential difference is given by the on (1).

adrant Electrometer. The effect measured by the instru-st described varies as the square of the potential difference; en the potential difference is diminished the attraction between es diminishes with great rapidity. For this reason the instru-s not suited for the measurement of very small potential ces. To measure these another electrometer, also due to Kelvin, called the quadrant electrometer, is frequently em-

s instrument is represented in Fig. 30; it consists of a cage, y the four quadrants A, B, C, D; each quadrant is supported insulating stem, while the e quadrants A and C are ed by a metal wire, as are and D; thus A and C are at the same potential and are B and D. Each pair of nts is in connection with an le, E or F, by means of it can easily be put in c connection with any body the case of the instrument. the quadrants and insulated hem is a flat piece of alu-n shaped like a figure of eight. suspended by a silk fibre n rotate, with its plane tal, about a vertical axis. metal wire hangs from the surface of this aluminium and dips into some sulphuric

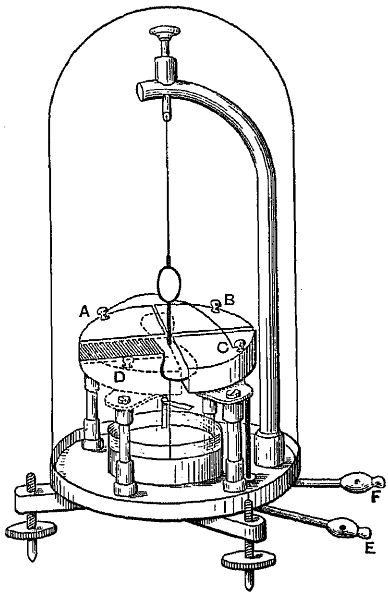


Fig. 30

ntained in a glass vessel, the outside of which is coated with and connected with earth. This vessel, with the conductors and outside, forms a condenser of considerable capacity; it

requires therefore a large charge to alter appreciably the potential of this jar, and therefore of the needle. To use the instrument the jar is charged to a high potential C; the needle will then also be at the potential C. Now if the two pairs of quadrants are at the same potential, the needle is inside a conductor symmetrical about the axis of rotation of the needle, and at one potential. There will evidently be no couple on the needle arising from the electric field, and the needle will take up a position in which the couple arising from the torsion of the thread supporting the needle vanishes. If, however, the two pairs of quadrants are not at the same potential the needle will swing round until, if there is nothing to stop it, the whole of its area will be inside the pair of quadrants whose potential differs most widely from its own. As it swings round, however, the torsion of the thread produces a couple tending to bring the needle back to the position from which it started. The needle finally takes up a position in which the couple due to the torsion in the thread balances that due to the electric field. The angle through which the needle is deflected gives us the means of estimating the potential difference between the quadrants.

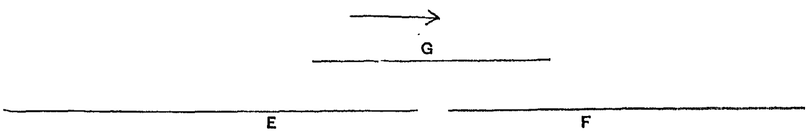


Fig. 31

The way in which the couple acting on the needle depends upon the potentials of the quadrants and the needle can be illustrated by considering a case in which the electric principles involved are the same as in the quadrant electrometer, but where the geometry is simpler.

Let E, F (Fig. 31) be two large co-planar surfaces insulated from each other by a small air-gap. Let G be another plane surface, parallel to E and F, and free to move in its own plane. Let t be the distance between G and the planes E and F. Let A, B, C be the potentials of the planes F, E, G respectively. Let l be the width of the planes at right angles to the plane of the paper. If Xl is the force tending to move the plane G in the direction of the arrow, then, if this plane be moved through a short distance x in this direction, the

work done by the electric forces is Xlx. If the electric system is left to itself, i.e. if it is not connected to any batteries, etc., so that the charges remain constant, this work must have been gained at the expense of the electric energy; we have therefore, by the principle of the Conservation of Energy,

$Xlx =$ *decrease* in the electric energy of the system, the charges remaining constant, when the plane G is displaced through the distance x;

or by Art. 36,

$Xlx =$ *increase* in the electric energy of the system, the potentials remaining constant, when the plane G is displaced through the same distance x ....(1).

Consider the change in the electric energy when the plane G is noved through a distance x. The area of G opposite to F will be ncreased by lx, and in consequence the energy will be increased by he energy in a parallel plate condenser, whose area is lx, the poten ials of whose plates are A and C respectively, and the distance etween the plates is t; this, by Art. 56, is equal to

$$\frac{lx}{8\pi t}(C - A)^2.$$

At the same time as the area of G opposite to F is increased by lx, hat opposite to E is decreased by the same amount, so that the lectric energy will be decreased by the energy in a parallel plate ondenser whose area is lx, the potentials of the plates B and C and heir distance apart t; this, by Art. 56, is equal to

$$\frac{lx}{8\pi t}(C - B)^2.$$

Thus the total increase in the electric energy when G is displaced hrough x, the potentials being constant, is equal to

$$\frac{1}{8\pi}\frac{lx}{t}\{(C - A)^2 - (C - B)^2\} = \frac{1}{4\pi}\frac{lx}{t}(B - A)\{C - \tfrac{1}{2}(A + B)\}.$$

Thus, by equation (1),

$$Xlx = \frac{1}{4\pi}\frac{lx}{t}(B - A)\{C - \tfrac{1}{2}(A + B)\},$$

$$X = \frac{1}{4\pi t}(B - A)\{C - \tfrac{1}{2}(A + B)\}.$$

If $(C - A)^2$ is greater than $(C - B)^2$, X is positive, that is, the plate G tends to bring as much of its surface as it can over the plate from which it differs most in potential.

In the quadrant electrometer the electrical arrangements are similar to the simple case just discussed, and hence the force will vary with the potential differences in a similar way. Hence we conclude that if the needle in the quadrant electrometer be at potential C, the couple tending to twist it from the quadrant whose potential is B to that whose potential is A, will be proportional to

$$(B - A)\left\{C - \frac{1}{2}(A + B)\right\};$$

we may put it equal to

$$n(B - A)\left\{C - \frac{1}{2}(A + B)\right\},$$

where n is some constant.

When the needle is in equilibrium, this couple will be balanced by the couple due to the torsion in the suspension of the needle.

The torsional couple is proportional to the angle θ through which the needle is deflected. Let the couple equal $m\theta$. Hence we have when the needle is in equilibrium

$$m\theta = n(B - A)\left\{C - \frac{1}{2}(A + B)\right\},$$

$$\theta = \frac{n}{m}(B - A)\left\{C - \frac{1}{2}(A + B)\right\} \quad \text{...........(2).}$$

If, as is generally the case when small differences of potential are measured, the jar containing the sulphuric acid is charged up so that its potential is very high compared with that of either pair of quadrants, C will be very large compared with A or B, and therefore with

$$\frac{1}{2}(A + B),$$

so that the expression (2) is very approximately

$$\theta = \frac{n}{m}(B - A)\,C.$$

Hence, in this case, the difference of potential is proportional to the deflection of the needle. This furnishes a very convenient method of comparing differences of potential, and though it does not give

he absolute measure of the potential, this may be deduced
suring the deflection produced by a standard potential
e of known absolute value such as that between the elec-
f a Clark's cell.

quadrant electrometer may also be used to measure large
es of potential; to do this, instead of charging the jar inde-
y, connect the jar and therefore the needle to one pair of
ts, say the pair whose potential is A. Then, since $C = A$, the
n (2) becomes

$$\theta = -\frac{n}{2m}(A - B)^2;$$

needle is deflected towards the pair of quadrants whose
is B, and the deflection of the needle is, in this case,
onal to the *square* of the potential difference between the
ts. Thus, if the quadrants are connected respectively
nside and outside coatings of a condenser, the deflection
electrometer will be proportional to the energy in the
er.

her method of using the electrometer is to connect the needle
ody whose potential is to be measured and keep the quad-
a known difference of potentials, by altering this difference
ange of potentials can be measured.

ype of quadrant electrometer now most frequently used and
s the Dolezalek electrometer differs in some important details
original type. In the first place the quadrants are much
and since the deflection of an electrometer varies inversely
distance of the needle from the quadrants, the diminution
istance increases the sensitiveness. Secondly, the needle is
silvered paper and is exceedingly light so that a very fine
on is sufficient to sustain its weight; with these fine suspen-
value of m is much reduced and this increases the sensitive-
this type of instrument the jar with sulphuric acid is dis-
vith and the potential of the needle is kept constant by put-
n electrical connection with a source of constant potential
a battery of small storage cells, the needle is suspended by
of thin metal strip so as to put it into connection with the
It is easy with an electrometer of this kind to get deflections

of the spot of light reflected from the mirror at the rate per volt of 5000 millimetres at a distance of 1 metre with a potential of the needle not exceeding 200 volts.

61. Use of the Electrometer to measure a charge of electricity. Let α and β denote the two pairs of quadrants. If to begin with α and β are both connected with the earth, there will be a charge Q_0 on the quadrants α induced by the charge on the needle; let α now be disconnected from β and from the earth, insulated, and given a charge Q' of electricity, the needle will be deflected; let θ be the angle of deflection, A the potential of the quadrants α, then if C is the potential of the needle, we have, by Art. 26, since the charge on α is $Q_0 + Q'$

$$Q_0 + Q' = q_{11}A + q_{13}C \quad \text{....................(1)},$$

where q_{11}, q_{13} are the coefficients of capacity and induction for the displaced position of the needle. Since Q_0 is the charge on α when A is zero

$$Q_0 = (q_{13})_0 C,$$

where $(q_{13})_0$ is the value of q_{13} when $\theta = 0$; hence by (1)

$$Q' = q_{11}A + (q_{13} - (q_{13})_0) C.$$

Let
$$q_{13} - (q_{13})_0 = -\mu\theta,$$

θ being taken as positive when measured in the direction of deflection due to a positive value of A, then if the charge on the needle is negative Q_0 the positive charge on α induced by the needle will evidently increase with θ so that as C is negative μ is a positive quantity; we have also by equation (2), page 74, when C is large compared with A,

$$\theta = -\frac{n}{m} AC,$$

hence
$$Q' = -\theta \left\{\frac{q_{11}m}{nC} + \mu C\right\},$$

or
$$\theta = -\frac{Q'nC}{q_{11}m + \mu nC^2} \quad \text{......................(2)}.$$

It is interesting to notice that when the potential of the needle is increased beyond a certain point the deflection of the needle due to a given charge on the quadrants diminishes as the potential of the needle increases, hence to obtain the greatest sensitiveness when

suring electrical charges we must be careful not to charge the
dle too highly. We see from (2) that the greatest deflection θ'
to the charge Q' is given by the equation

$$\theta' = \tfrac{1}{2}Q'\sqrt{\frac{n}{\mu m q_{11}}};$$

n the deflection is greatest the potential of the needle

$$= (q_{11}m/\mu n)^{\frac{1}{2}}.$$

o get from the readings of the electrometer the value of the
ge in absolute measure, connect one plate of a condenser whose
city is Γ with the quadrants a, and connect the other plate with
earth; the coefficient q_{11} will now be increased by Γ and, if θ_1 is
deflection of the electrometer for the same charge, then by (2)

$$\theta_1 = \frac{Q'nC}{(q_{11}+\Gamma)m + \mu nC^2} \quad\text{.................. (3).}$$

e from (2) and (3)

$$\theta = \frac{\theta - \theta_1}{\theta_1}\frac{Q'nC}{\Gamma m} \quad\text{.................. (4).}$$

the deflection of the electrometer is ϕ when the potential of a
then

$$\phi = \frac{n}{m}CV,$$

, from (4),

$$Q' = \Gamma V\frac{\theta}{\phi}\cdot\frac{\theta_1}{\theta-\theta_1}.$$

he insulation of the quadrants is of primary importance when
es of electricity have to be measured.

a. A gold leaf electroscope is for some purposes preferable
electrometer, on account of its much smaller capacity, its
bility and the ease with which it can be shielded from external
bances. With suitably designed electroscopes it is possible to
with ease a deflection of the gold leaf of 70 or 80 scale divisions
hange of 1 volt in the potential of the gold leaf, these divisions
ose of a micrometer eye piece in a reading microscope through
the gold leaf is observed. The behaviour of these sensitive
oscopes may be illustrated by the consideration of a very
case. Suppose that we have two parallel plates D and E

the gold leaf by another parallel plate C which can move backwards and forwards and is pulled to a position midway between D and E by a spring, which when C is displaced a distance x from the mid-position pulls it back again with a force equal per unit area of C to μx.

If V is the potential of C, $2d$ the distance between D and E, x the displacement of C towards the negative plate, then for the equilibrium of the plate we must have

$$\frac{1}{8\pi}\frac{(A+V)^2}{(d-x)^2} - \frac{1}{8\pi}\frac{(A-V)^2}{(d+x)^2} = \mu x,$$

or if $V = yA$, $x = \xi d$,

$$\frac{(1+y)^2}{(1-\xi)^2} - \frac{(1-y)^2}{(1+\xi)^2} = \frac{8\pi\mu d^3}{A^2}\xi = 4\mu'\xi$$

if

$$\mu' = \frac{2\pi\mu d^3}{A^2}.$$

If y and ξ are small, this equation becomes

$$(\mu' - 1)\,\xi = y,$$

or

$$x = \frac{d\,.\,\dfrac{V}{A}}{\dfrac{2\pi\mu d^3}{A^2} - 1}.$$

The equilibrium will be unstable unless $2\pi\mu d^3/A^2$ is greater than 1, when this quantity exceeds unity by a small fraction the denominator in the expression for x is small so that x itself tends to become large, i.e. a small potential difference V will produce a large displacement of the plate. In addition to the value of x given above there is a second value corresponding to another position of equilibrium, the equilibrium in this case is unstable and if C were in this position it would move up to D. When $V = 0$ the two positions of equilibrium are given by $x = 0$ and

$$\frac{x^2}{d^2} = 1 - \frac{1}{\sqrt{\mu'}}.$$

As V increases the value of x for the stable position increases while that for the unstable one diminishes, so that the two get nearer together, for a certain value of V they coincide, while for greater

there is no position of equilibrium. When the instrument is ensitive, i.e. when μ' is nearly unity, the second value of x is ry small when V is small, thus the unstable position of equilib- close to the stable one, so that a slight deflection from the latter ıke the gold leaf unstable, and it will fly up to one of the plates. practice the office of the spring in the preceding example is ned by the weight of the gold leaf; the leaf is hung so as to be l when midway between the plates, when it is disturbed from sition gravity tends to bring it back. The successful use of nents of this type depends upon having means to keep the

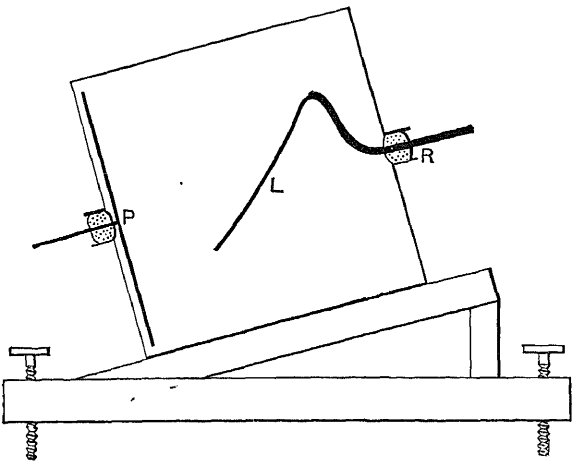


Fig. 31 *a*

ial of the fixed plates accurately constant. Except for very values of V, the deflection is not directly proportional to V, t it is necessary to calibrate the instrument by charging the af to known potentials and observing the deflection.

e sensitiveness of the instrument can be adjusted by altering In a type of instrument invented by Mr C. T. R. Wilson and the tilted electroscope (Fig. 31 *a*), where the instrument can ed by means of foot-screws, the adjustment is effected by g the tilt. The plate P is charged to a high potential, the case instrument to earth, and initially the gold leaf is to earth, s up a position of equilibrium from which it is displaced as s its potential is altered.

62. Test for the equality of the capacities of two condensers. The test can easily be made in the following way. Suppose A and B, Fig. 32, are the plates of one condenser, C and D those of the other. First connect A to C, and B to D, and charge the condensers by connecting A and B with the terminals of a battery or some other suitable means. Then disconnect A and B from the battery. Disconnect A from C and B from D. Then, if the capacities of the two condensers are equal, their charges will be equal since they have been charged to equal potentials. The charge on A will be equal and opposite to that on D, while that on B will be equal and opposite to that on C. Thus, if A be connected with D and C with B, the positive charge on the one plate will counterbalance the negative on the other, so that if after this connection has been made A and B are connected with the electrodes of an electrometer, no deflection will occur.

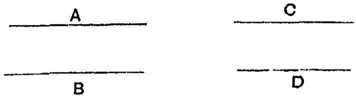


Fig. 32

63. Comparison of two condensers. If a condenser whose capacity can be varied is available, the capacity of a condenser can be compared with known capacities by the following method.

Let A and B (Fig. 33) be the plates of the condenser whose capacity is required, C and D, E and F, G and H, the plates of three condensers whose capacities are known. Connect the plates B and C together and to one electrode of an electrometer, also connect F and G together and to the other electrode of the electrometer. Connect D and E together and to one pole of a battery, induction coil or other apparatus for producing a difference of potential, and connect A and H together and to the other pole of this battery. In general this will cause a deflection of the electrometer; if there is a deflection, then we must alter the capacity of the condenser whose capacity is variable until the vanishing of this deflection shows that the plates BC, FG are at the same potential. When this is the case a simple relation exists between the capacities.

Let C_1, C_2, C_3, C_4 be the capacities of the condensers AB, CD, EF, GH respectively, let V_0 be the potential of A and H, x the potential of B and C and y that of F and G, V the potential of D and E. To fix our ideas, let us suppose that V is greater than V_0, then there

a negative charge on A, a positive one on B, a negative charge nd a positive one on D; then since B and C form an insulated which was initially without charge, the positive charge on B e numerically equal to the negative charge on C.

positive charge on B

$$= C_1 (x - V_0),$$

ne negative one on C is numerically equal to

$$C_2 (V - x),$$

s a positive quantity; hence, since these are equal, we have

$$C_1 (x - V_0) = C_2 (V - x) \quad\text{..................(1)}.$$

in, since F and G are insulated the positive charge on G must erically equal to the negative charge on F.

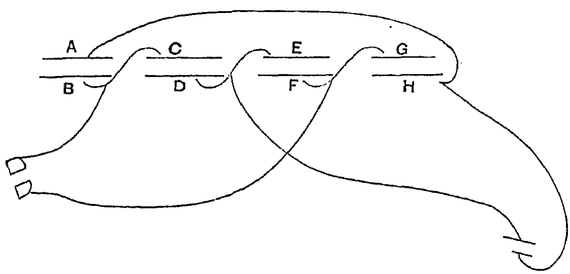


Fig. 33

positive charge on G is equal to

$$C_4 (y - V_0),$$

ne negative charge on F is numerically equal to

$$C_3 (V - y);$$

ese are equal

$$C_4 (y - V_0) = C_3 (V - y) \quad\text{................(2)}.$$

here is no deflection of the electrometer the potential of F and al to that of B and C, i.e. $y = x$. When this is the case we see paring equations (1) and (2), that

$$\frac{C_1}{C_4} = \frac{C_2}{C_3},$$

$$C_1 = \frac{C_2 C_4}{C_3}.$$

if we know the capacities of the other condensers, we know C_1.

Thus, if we have standard condensers whose capacities are known, we can measure the capacity of other condensers.

There is a close analogy between the methods of measuring capacity and those of measuring electrical resistance. It is convenient to indicate that analogy here, although the methods of measuring electrical resistance have not yet been discussed.

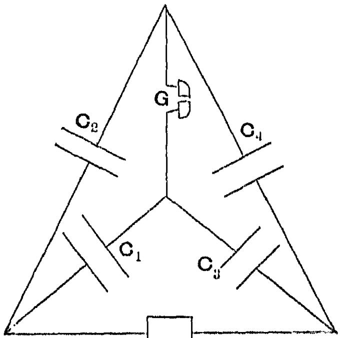


Fig. 34

The arrangement of the condensers in the last method can also be represented by the diagram (Fig. 34). In this diagram C is the coil and G the electrometer. This arrangement is analogous to that of resistances in a Wheatstone's Bridge, see Art. 191, and the condition for the balance of the condensers is the same as that of resistances in a Wheatstone bridge if each condenser were replaced by a resistance inversely proportional to its capacity.

63 *a*. De Sauty's method. If two of the condensers C_3 and C_4 in the last method are replaced by resistances R_3 and R_4, the electrometer by a galvanometer and the induction coil by a battery with a key for making and breaking the circuit, we get the arrangement known as De Sauty's method, Fig. 35.

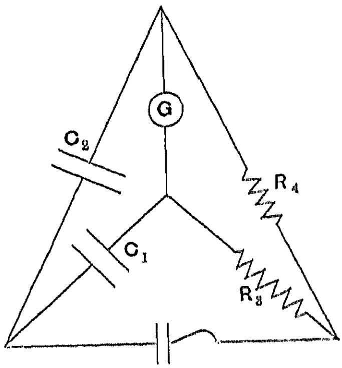


Fig. 35

In this method the resistances R_3 and R_4 are adjusted so that there is no kick of the galvanometer on making the battery circuit. If i_3 and i_4 are the transient currents flowing through R_3 and R_4 at some short interval after making the circuit, then neglecting self-induction, the potential difference at this time between the terminals of the galvanometer will, by Ohm's law, be $R_3 i_3 - R_4 i_4$, and this will be proportional to the current through the galvanometer at this time.

uantity of electricity flowing through the galvanometer during ng will thus be proportional to $\int(R_3i_3 - R_4i_4)\,dt$, when the ation extends over the time of charging. If no current flows gh the galvanometer, the current i_3 goes into the condenser d i_4 into condenser (2), so that

$$\int i_3\,dt = Q_1, \quad \int i_4\,dt = Q_2,$$

Q_1 and Q_2 are the final charges in condensers (1) and (2) re- vely. Thus

$$\int(R_3i_3 - R_4i_4)\,dt = R_3Q_1 - R_4Q_2,$$

there is no kick of the galvanometer this vanishes, so that

$$R_3Q_1 = R_4Q_2.$$

when the condensers are charged there is the same potential ence between the plates of (1) as between those of (2), hence

$$Q_1 : Q_2 = C_1 : C_2,$$

C_1, C_2 are the capacities of the condensers, hence when there kick of the galvanometer

$$R_3C_1 = R_4C_2,$$

the ratio C_1/C_2 is found as the ratio of two resistances. We see again the condition is the same as for the balance in a Wheat- bridge in which the condensers have been replaced by resist- inversely proportional to their capacity.

ther methods of determining capacity which require for their nation a knowledge of the principles of electro-magnetism, will scribed in the part of the book dealing with that subject.

4. **Leyden jar.** A convenient form of condenser called a en jar is represented in Fig. 36. ondenser consists of a vessel made in glass; the inside and outside ces of this vessel are coated with il. An electrode is connected to nside of the jar in order that elec- connection can easily be made with A is the area of each coat of tin- the thickness of the glass, i.e. the nce between the surfaces of tin-foil,

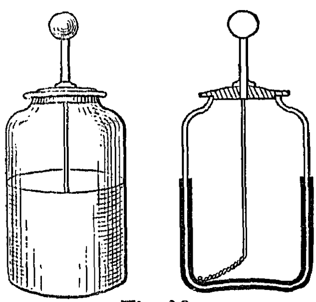


Fig. 36

then, if the interval between these surfaces was filled with air, the capacity would be approximately

$$\frac{A}{4\pi t},$$

since this case is approximately that of two parallel planes provided the thickness of the glass is very small compared with the dimensions of the vessel. The effect of having glass within the tin-foil surfaces will, as we shall see in the next chapter, have the effect of increasing the capacity so that the capacity of the Leyden jar will be

$$K\frac{A}{4\pi t},$$

where K is a quantity which depends on the kind of glass of which the vessel is made. K varies in value from 4 to 10 for different specimens of glass.

Systems of Condensers.

65. If we have a number of condensers we can connect them up so as to make a condenser whose capacity is either greater or less than that of the individual condensers.

Thus suppose we have a number of condensers which in the figures are represented as Leyden jars, and suppose we connect them up as in Fig. 37, that is, connect all the insides of the jars together and likewise all the outsides; this is called connecting the condensers in parallel. We thus get a new condenser, one plate of which consists of all the insides, and the other plate of all the outsides of the jars. If C is the capacity of the compound condenser, Q the total charge in this condenser, V the difference of potential between the plates, then by definition

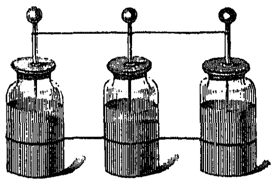


Fig. 37

$$Q = CV.$$

If Q_1, Q_2, Q_3, ... are the charges in the first, second, third, etc. condensers, C_1, C_2, C_3, ... the capacities of these condensers

$$Q_1 = C_1V, \quad Q_2 = C_2V, \quad Q_3 = C_3V, \text{ etc.};$$

but $$Q = Q_1 + Q_2 + Q_3 + \dots = (C_1 + C_2 + C_3 + \dots)V,$$

hence $$C = C_1 + C_2 + C_3 + \dots,$$

or the capacity of a system of condensers connected in this way, is

sum of the capacities of its components. Thus the capacity of compound system is greater than that of any of its components.

Next, let the condensers be connected up as in Fig. 38, where the densers are insulated, and where the side of the first is connected to the in- e of the second, the outside of the ond to the inside of the third, and so This is called connecting the condensers in cascade or in series. One plate of compound system thus formed is the

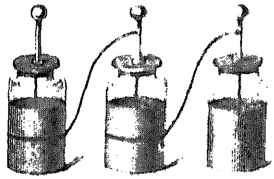


Fig. 38

de of the first condenser, the other plate is the outside of the last.

Let C be the capacity of the system, C_1, C_2, C_3, ... the capacities he individual condensers; then, since the condensers are insulated, charge on the outside of the first is equal in magnitude and oppo in sign to the charge on the inside of the second, the charge on outside of the second is equal in magnitude and opposite in sign he charge on the inside of the third, and so on. Since the charge he inside of any jar is equal and opposite to the charge on the ide, we see that the charges of the jars are all equal. Let Q be the ge of any jar, $V_1, V_2, \ldots$ the differences of potential between the le and outside of the first, second, ... jars. Then

$$V_1 = \frac{Q}{C_1}, \quad V_2 = \frac{Q}{C_2}, \quad V_3 = \frac{Q}{C_3}, \ldots$$

is the difference of potential between the outside of the last jar the inside of the first, then

$$V = V_1 + V_2 + V_3 \ldots$$

$$= Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots\right).$$

at

$$Q = \frac{V}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots}.$$

since C is the capacity of the compound condenser of which Q charge, and V the potential difference,

$$Q = CV,$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots.$$

Thus the reciprocal of the capacity of the system made by connecting up in cascade the series of condensers, is equal to the sum of the reciprocals of the capacities of the condensers so connected up.

We see that the capacity of the compound condenser is less than that of any of its constituents.

66. If we connect a condenser of small capacity in cascade with a condenser of large capacity, the capacity of the compound condenser will be slightly less than that of the small condenser; while if we connect them in parallel, the capacity of the compound condenser is slightly greater than that of the large condenser.

67. As another example on the theory of condensers, let us take the case when two condensers are connected in parallel, the first having before connection the charge Q_1, the second the charge Q_2. Let C_1 and C_2 be the capacities of these condensers respectively. When they are put in connection they form a condenser whose capacity is $C_1 + C_2$, and whose charge is $Q_1 + Q_2$.

Now the electric energy of a charged condenser is one half the product of the charge into the potential difference, while the potential difference is equal to the charge divided by the capacity. Thus if Q is the charge, C the capacity, the energy is

$$\frac{1}{2}\frac{Q^2}{C}.$$

Thus the total electric energy of the two jars before they are connected is

$$\frac{1}{2}\frac{Q_1^2}{C_1} + \frac{1}{2}\frac{Q_2^2}{C_2};$$

after they are connected it is

$$\frac{1}{2}\frac{(Q_1 + Q_2)^2}{C_1 + C_2};$$

Now

$$\frac{1}{2}\left(\frac{Q_1^2}{C_1} + \frac{Q_2^2}{C_2}\right) - \frac{1}{2}\frac{(Q_1 + Q_2)^2}{(C_1 + C_2)}$$

$$= \frac{1}{2C_1C_2(C_1 + C_2)}(C_2^2Q_1^2 + C_1^2Q_2^2 - 2C_1C_2Q_1Q_2)$$

$$= \frac{1}{2C_1C_2(C_1 + C_2)}(C_2Q_1 - C_1Q_2)^2,$$

ntially positive quantity which only vanishes if

$$Q_1/C_1 = Q_2/C_2,$$

when the potentials of the jars before connection are equal. case the energy after connection is the same as before the ions are made. If the potentials are equal before connection, ing the jars will evidently make no difference, as all that ion does is to make the potentials equal. In every other case energy is lost when the connection is made; this energy is ed for by the work done by the spark which passes when the connected.

CHAPTER IV

SPECIFIC INDUCTIVE CAPAC

68. Specific Inductive Capacity. Far charge in a condenser between whose surfaces a c potential was maintained depended upon the nat between the surfaces, the charge being greater between the surfaces was filled with glass or su was filled with air.

Thus the 'capacity' of a condenser (see Art. 51 dielectric between the plates. Faraday's origi which this result was establisl he took two equal and similar c of the kind shown in Fig. 39, spheres; in one of these, B, th by which melted wax or sulphu the interval between the sphe these condensers were connecte also the outsides, so that the between the plates of the cond for A as for B. When air was th the spheres Faraday found, as r pected from the equality of tl any charge given to the cond distributed between A and B.

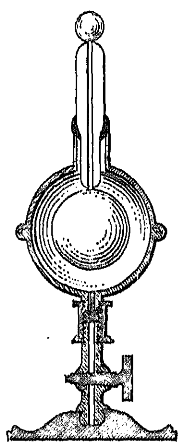


Fig. 39

interval in B was filled with sul densers again charged he found that the charge four times that in A, proving that the capaci increased three or four times by the substitution

This property of the dielectric is called its capacity. The measure of the specific inductive c tric is defined as the ratio of the capacity of a c region between its plates is entirely filled by th capacity of the same condenser, when the region is entirely filled with air. As far as we know at p

luctive capacity of a dielectric in a condenser does not depend on the difference of potential established between the plates of it condenser, that is, upon the electric intensity acting on the lectric. We may therefore conclude that, at any rate for a wide ge of electric intensities, the specific inductive capacity is inde ident of the electric intensity.

The following table contains the values of the specific inductive acities of some substances which are frequently used in a physical oratory:

Solid paraffin	2·29.
Paraffin oil	1·92.
Ebonite	3·15.
Sulphur	3·97
Mica	6·64.
Dense flint glass	7·37.
Light flint glass	6·72.
Turpentine	2·23.
Distilled water	76.
Alcohol	26.

The specific inductive capacity of gases depends upon the pres , the difference between K, the specific inductive capacity, and y being directly proportional to the pressure.

The specific inductive capacity of some gases at atmospheric sure is given in the following table; the specific inductive capacity r at atmospheric pressure is taken as unity:

Hydrogen	·99967
Carbonic acid	1·000356
Carbonic oxide	1·0001
Olefiant gas	1·000722
Water vapour (at 140° C)	1·00705
Methyl alcohol vapour (at 110° C)	1·0057

hus we see that water and alcohol possess abnormally high ic inductive capacities both in the liquid and gaseous states.

9. It was the discovery of this property of the dielectric which araday to the view we have explained, in Art. 38, that the effects ved in the electric field are not due to the action at a distance

of one electrified body on another, but are due to effects in the dielectric filling the space between the electrified bodies.

The results obtained in Chapters II and III were deduced on the supposition that there was only one dielectric, air, in the field; these require modification in the general case when we have any number of dielectrics in the field. We shall now go on to consider the theory of this general case.

We assume that each unit of positive electricity, whatever be the medium by which it is surrounded, is the origin of a Faraday tube, each unit of negative electricity the termination of one. Let us consider from this point of view the case of two parallel plate condensers A and B, the plates of A and B being at the same distance apart, but while the plates of A are separated by air, those of B are separated by a medium whose specific inductive capacity is K. Let us suppose that the charge per unit area on the plates of the condensers A and B is the same. Then, since the capacity of the condenser B is K times that of A and since the charges are equal, the potential difference between the plates of B is only $1/K$ of that between the plates of A.

Now if V_P is the potential at P, V_Q that at Q, R the electric intensity along PQ, then, whatever be the nature of the dielectric, when PQ is small enough to allow of the intensity along it being regarded as constant,

$$R \cdot PQ = V_P - V_Q \quad \text{..................(1)},$$

for by definition R is the force on unit charge, hence the left-hand side of this expression is the work done on unit charge as it moves from P to Q, and is thus by definition (Art. 16), equal to the right-hand side of (1).

The electric intensity between the plates both of A and of B is uniform, and is equal to the difference of potential between the plates divided by the distance between the plates; this distance is the same for the plates A and B, so that the electric intensity between the plates of A is to that between the plates of B as the potential difference between the plates of A is to that between the plates of B. That is, the electric intensity in A is K times that in B.

Consider now these two condensers. Since the charges on unit area of the plates are equal the number of Faraday tubes passing through the dielectric between the plates is the same, while the electric intensity in B is only $1/K$ that in air. Hence we conclude

hat when the number of Faraday tubes which pass through unit rea of a dielectric whose specific inductive capacity is K is the same s the number which pass through unit area in air, the electric in ensity in the dielectric is $1/K$ of the electric intensity in air.

By Art. 40, we see that if N is the number of Faraday tubes assing through unit area in air, and R is the electric intensity in air,

$$R = 4\pi N.$$

ence, when N tubes pass through unit area in a medium whose ecific inductive capacity is K, the electric intensity, R, in this electric is given by the equation

$$R = \frac{4\pi}{K} N.$$

70. Polarization in a dielectric. We define the polariza n in the direction PQ where P and Q are two points close together the excess of number of Faraday tubes which pass from the side o the side Q over the number which pass from the side Q to the e P of a plane of unit area drawn between P and Q at right angles PQ. We may express the result in Art. 68 in the form

electric intensity in any direction at P)

$$= \frac{4\pi}{K} \text{(polarization in the dielectric in that direction at } P)$$

The polarization in a dielectric is mathematically identical with quantity called by Maxwell the electric displacement in the ectric.

71. Thus the polarization along the outward drawn normal at P surface is the excess of the number of Faraday tubes which leave surface through unit area at P over the number entering it. e divide any closed surface up as in Art. 9 into a number of small hes, each of these meshes being so small that the polarization the area of any mesh may be regarded as constant, then if we iply the area of each of the meshes by the normal polarization is mesh measured outwards, the sum of the products taken for he meshes which cover the surface is defined to be the total al polarization outwards from the surface. We see that it is l to the excess of the number of Faraday tubes which leave the ce over the number which enter it.

Now consider any tube which does not begin or end inside the closed surface, then if it meets the surface at all it will do so at two places, P and Q; at one of these it will be going from the inside to the outside of the surface, at the other from the outside to the inside. Such a tube will not contribute anything to the total normal polarization outwards from the surface, for at the place where it leaves the surface it contributes $+1$ to this quantity, which is neutralized by the -1 which it contributes at the place where it enters the surface.

Now consider a tube starting inside the surface; this tube will leave the surface but not enter it, or if the surface is bent so that the tube cuts the surface more than once, it will leave the surface once oftener than it enters it. This tube will therefore contribute $+1$ to the total outward normal polarization: similarly we may show that each tube which ends inside the surface contributes -1 to the total outward normal polarization. Thus if there are N tubes which begin, and M tubes which end inside the surface, the total normal polarization is equal to $N-M$. But each tube which begins inside the surface corresponds to a unit positive charge, each tube which ends in the surface to a unit negative one, so that $N-M$ is the difference between the positive and negative charges inside the surface, that is, it is the total charge inside the surface.

Thus we see that the total normal polarization over a closed surface is equal to the charge inside the surface. Since the normal polarization is equal to $K/4\pi$ times the normal intensity where K is the specific inductive capacity, which is equal to unity for air, we see that when the dielectric is air the preceding theorem is identical with Gauss's theorem, Art. 10. In the form stated above it is applicable whatever dielectrics may be in the field, when in general Gauss's theorem as stated in Art. 10 ceases to be true.

72. Modification of Coulomb's equation. If σ is the surface density of the electricity on a conductor, then σ Faraday tubes pass through unit area of a plane drawn in the dielectric just above the conductor at right angles to the normal. Hence σ is the polarization in the dielectric in the direction of the normal to the conductor. Hence, by Art. 69, if R is the normal electric intensity

$$R = \frac{4\pi}{K}\sigma.$$

This is Coulomb's equation generalized so as to apply to the case en the conductor is in contact with any dielectric.

73. Expression for the Energy. The student will see that process of Art. 23 by which the expression $\frac{1}{2}\Sigma(EV)$ was proved represent the electric energy of the system will apply whatever nature of the dielectric may be, as will also the immediate deduc from it in Art. 43 that the energy is the same as it would be if h Faraday tube possessed an amount of energy equal per unit gth to one-half the electric intensity.

The expression for the energy per unit volume however requires dification. Consider, as in Art. 43, a cylinder whose axis is parallel the electric intensity and whose flat ends are at right angles to it, l be the length of the cylinder, ω the area of one of the ends, P the arization, R the electric intensity. Then the portion of each Fara tube inside the cylinder has an amount of energy equal to

$$\tfrac{1}{2}lR.$$

v the number of such tubes inside the cylinder is equal to $P\omega$, ce the energy inside the cylinder is equal to

$$\tfrac{1}{2}l\omega PR.$$

ce $l\omega$ is the volume of the cylinder, the energy per unit volume qual to

$$\tfrac{1}{2}PR;$$

by Art. 69 $$P = \frac{K}{4\pi}R,$$

hat the energy per unit volume is equal to

$$\frac{K}{8\pi}R^2.$$

Thus, for the same electric intensity the energy per unit volume ne dielectric is K times as great as it is in air. Another expression he energy per unit volume is

$$\frac{2\pi}{K}P^2,$$

hat for same polarization the energy per unit volume in the ctric is only $1/K$th part of what it is in air.

We see, as in Art. 45, that the pull along each Faraday tube will still equal one-half the electric intensity R; the tension across unit area in the dielectric will therefore be $\frac{KR^2}{8\pi}$, the lateral pressure will also be equal to $KR^2/8\pi$.

74. Conditions to be satisfied at the boundary between two media of different specific inductive capacities. Suppose that the line AB represents the section by the plane of the paper of the plane of separation between two different dielectrics; let the specific inductive capacities of the upper and lower media respectively be K_1, K_2.

Let us consider the conditions which must hold at the surface. In the first place we see that the electric intensities parallel to the surface must be equal in the two media; for if they were not equal, and that in the medium K_1 were the greater, we could get an infinite amount of work by making unit charge travel round the closed circuit $PQRS$, PQ being just above, and RS just below the surface of separation. For, if PQ is the direction of T_1 the tangential component of the electric intensity in the upper medium, the work done on unit charge as it goes from P to Q is $T_1 . PQ$; as QR is exceedingly small compared with PQ the work done on or by the charge as it goes from Q to R may be neglected if the normal intensity is not infinite; the work required to take the unit charge back from R to S is $T_2 . RS$, if T_2 is the tangential component of the electric intensity in the lower dielectric, and the work done or spent in going from S to P will be equal to that spent or done in going from Q to R and may be neglected. Thus since the system is brought back to the state from which it started, the work done must vanish, and hence $T_1 . PQ - T_2 . RS$ must be zero. But since $PQ = RS$ this requires that $T_1 = T_2$ or the tangential components of the electric intensity must be the same in the two media.

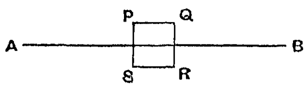


Fig. 40

Next suppose that σ is the surface density of the free electricity on the surface separating the two media. Draw a very flat circular cylinder shown in section at $PQRS$, its axis being parallel to the normal to the surface of separation, the top face of this

ylinder being just above, the lower face just below this surface. s the length of this cylinder is very small compared with its breadth, ne area of the curved surface of the cylinder will be very small ompared with the area of its ends, and by making the cylinder fficiently short we can make the ratio of the area of the curved rface to that of the ends as small as we please. Hence in con dering the total outward normal polarization over the very short linder, we may leave out the effect of the curved surface and con der only the flat ends of the cylinder. But since the cylinder closes the charge $\sigma\omega$, if ω is the area of one end of the cylinder, the tal normal polarization over its surface must be equal to $\sigma\omega$. If N_1 the normal polarization in the first medium measured upwards e total normal polarization over the top of the cylinder is $N_1\omega$; N_2 is the normal polarization measured upwards in the second edium, the total normal polarization over the lower face of the linder is $-N_2\omega$; hence the total outward normal polarization over e cylinder is

$$N_1\omega - N_2\omega.$$

Since, by Art. 71, this is equal to $\sigma\omega$, we have

$$N_1 - N_2 = \sigma.$$

When there is no charge on the surface separating the two di ctrics, these conditions become (1) that the tangential electric ensities, and (2) the normal polarizations, must be equal in the o media.

75. **Refraction of the lines of force.** Suppose that R_1 he resultant electric intensity in the upper medium, R_2 that in lower; and θ_1, θ_2 the angles these make with the normal to the face of separation. The tangential intensity in the first medium is sin θ_1, that in the second is $R_2 \sin\theta_2$, and since these are equal

$$R_1 \sin\theta_1 = R_2 \sin\theta_2 \quad \ldots \ldots \quad (1).$$

The normal intensity in the upper medium is $R_1 \cos\theta_1$, hence normal polarization in the upper medium is

$$K_1 R_1 \cos\theta_1/4\pi,$$

in the second is $K_2 R_2 \cos\theta_2/4\pi$, and since, if there is no charge he surface, these are equal, we have

$$\frac{K_1}{4\pi} R_1 \cos\theta_1 = \frac{K_2}{4\pi} R_2 \cos\theta_2 \quad \ldots \quad (2).$$

dividing (1) by (2), we get

$$\frac{1}{K_1}\tan\theta_1 = \frac{1}{K_2}\tan\theta_2.$$

Hence, if $K_1 > K_2$, θ_1 is $> \theta_2$, and thus when a Faraday tube enters a medium of greater specific inductive capacity from one of less, it is bent away from the normal.

This is shown in the diagram Fig. 41 (from Lord Kelvin's Reprint of Papers on Electrostatics and Magnetism), which represents the Faraday tubes when a sphere, made of paraffin or some material whose specific inductive capacity is greater than unity, is placed in a field of uniform force such as that between two infinite parallel plates.

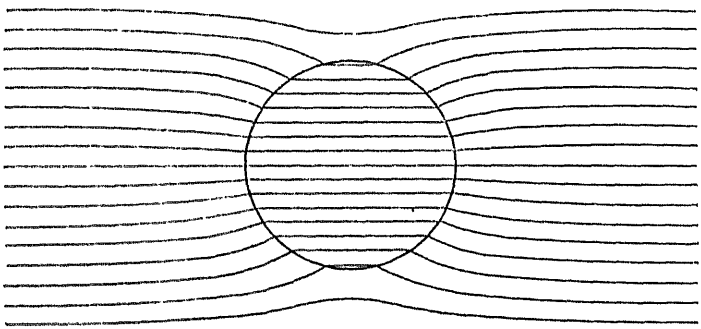


Fig. 41

An inspection of the diagram shows the tendency of the tubes to run as much as possible through the sphere; this is an example of the principle that when a system is in stable equilibrium the potential energy is as small as possible. We saw, Art. 73, that when the polarization is P the energy per unit volume is $2\pi P^2/K$, thus for the same value of P, this quantity is less in paraffin than it is in air. Hence when the same number of tubes pass through the paraffin they have less energy in unit volume than when they pass through air, and there is therefore a tendency for the tubes to flock into the paraffin. The reason why all the tubes do not run into the sphere is that those which are some distance away from it would have to bend considerably in order to reach the paraffin, they would therefore have to greatly lengthen their path in the air, and the increase in the

consequent upon this would not be compensated for in the f the tubes which were far from the sphere by the diminu- the energy when they got in the sphere.

Fig. 42 (from Lord Kelvin's Reprint of Papers on Electrostatics agnetism) the effects produced on a field of uniform force by a cting sphere is given for comparison with the effects produced paraffin sphere. It will be noticed that the paraffin sphere ces effects similar in kind though not so great in degree as those the conducting sphere. This observation is true for all electro- phenomena, for we find that bodies having a greater specific

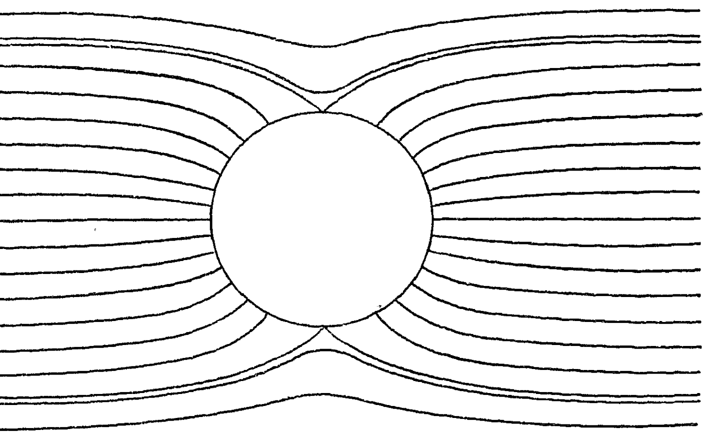


Fig. 42

tive capacity than the surrounding dielectric behave in a similar to conductors. Thus, they deflect the Faraday tubes in the way though not to the same extent; again, as a conductor tends ove from the weak to the strong parts of the field, so likewise a dielectric surrounded by one of smaller specific inductive city. Again, the electric intensity inside a conductor vanishes, ust inside a dielectric of greater specific inductive capacity than urrounding medium the electric intensity is less than that just de. As far as electrostatic phenomena are concerned an insulated uctor behaves like a dielectric of infinitely great specific in- ve capacity.

76. Force between two small charged bodies immersed in any dielectric. If we have a small body with a charge e immersed in a medium whose specific inductive capacity is K, then the polarization at a distance r from the body is $e/4\pi r^2$. To prove this, describe a sphere radius r, with its centre at the small body, then the polarization P will be uniform over the surface of the sphere and radial; hence the total normal polarization over the surface of the sphere will equal $P \times$ (surface of the sphere), i.e. $P \times 4\pi r^2$; but this, by Art. 71, is equal to e, hence

$$P \times 4\pi r^2 = e,$$

or
$$P = \frac{e}{4\pi r^2} \quad \text{.........................(1).}$$

But, if R is the electric intensity, then, by Art. 70,

$$R = \frac{4\pi}{K} \cdot P.$$

Hence, by (1), $$R = \frac{e}{Kr^2};$$

the repulsion on a charge e' is Re', or ee'/Kr^2; hence the repulsion between the charges, when separated by a distance r in a dielectric whose specific inductive capacity is K, is only $1/K$th part of the repulsion between the charges when they are separated by the same distance in air. Thus, when the charges are given, the mechanical forces on the bodies in the field are diminished when the charges are imbedded in a medium with a large specific inductive capacity. We can easily show that the interposition of a spherical shell of the dielectric with its centre at either of the charges would not affect the force between these charges.

77. Two parallel plates separated by a dielectric. Let us first take the case of two parallel plates completely immersed in an insulating medium whose specific inductive capacity is K. Let V be the potential difference between the plates, σ the surface density of the electrification on the positive plate, and $-\sigma$ that on the negative. Let R be the electric intensity between the plates, and d the distance by which they are separated; then, by Art. 72,

$$4\pi\sigma = KR$$
$$= \frac{KV}{d}.$$

The force on one of the plates per unit area is, by Art. 37,

$$\tfrac{1}{2}R\sigma$$
$$=\frac{2\pi\sigma^2}{K}.$$

[H]ence if the charges are given the force between the plates is *in[v]ersely* proportional to the specific inductive capacity of the medium [i]n which they are immersed.

Again, since $\dfrac{1}{2}R\sigma=\dfrac{1}{8\pi}KR^2=\dfrac{K}{8\pi}\dfrac{V^2}{d^2}$,

[w]e see that, if the potentials of the plates be given, the attraction [b]etween them is *directly* proportional to the specific inductive [c]apacity. This result is an example of the following more general [o]ne which we leave to the reader to work out; if in a system of con[d]uctors maintained at given potentials and originally separated from [e]ach other by air we replace the air by a dielectric whose specific [in]ductive capacity is K, keeping the positions of the conductors and [th]eir potentials the same as before, the forces between the conductors [w]ill be increased K times.

Thus, for example, if we fill the space between the needles and the [q]uadrants of an electrometer with a fluid whose specific inductive [ca]pacity is K, keeping the potentials of the needles and quadrants [co]nstant, the couple on the needle will be increased K times by the [in]troduction of the fluid. If we measure the couples before and after [th]e introduction of the fluid, the ratio of the two will give us the [sp]ecific inductive capacity of the fluid. This method has been applied [to] measure the specific inductive capacity of those liquids, such as [w]ater or alcohol, which are not sufficiently good insulators to allow [th]e method described in Art. 82 to be applied.

78. We shall next consider the case in which a slab of dielectric [is] placed between two infinite parallel conducting planes, the faces [of] the slab being parallel to the planes.

Let d be the distance between the planes, t the thickness of the [sla]b, h the distance between the upper face of the slab and the upper [pl]ane. The Faraday tubes will go straight across from plane to [pl]ane, so that the polarization will be everywhere normal to the con[du]cting planes and to the planes separating the slab of dielectric from [th]e air.

as we pass from one medium to another, and as the tubes are straight the polarization will not change as long as we remain in one medium. Thus the polarization which we shall denote by P is constant between the planes. In air the electric intensity is $4\pi P$; in the dielectric of specific inductive capacity K, the electric intensity is equal to $4\pi P/K$.

Thus between A and B the electric intensity is $4\pi P$,

.... B and C .... $\dfrac{4\pi P}{K}$,

.... C and D .... $4\pi P$.

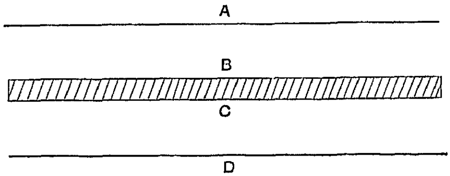


Fig. 43

The difference of potential between the plates is the work done on unit charge when it is taken from one plate to the other. Now, when unit charge is taken across the space AB, the work done on it is

$$4\pi P \times h;$$

when it is taken across the plate of dielectric the work done is

$$\frac{4\pi P}{K} \times t;$$

when it is taken across CD the work done is

$$4\pi P\,\{d - (h + t)\}.$$

Hence V, the excess of the potential of the plate A above that of D, is equal to

$$4\pi Ph + \frac{4\pi P}{K} t + 4\pi P\,\{d - (h + t)\}$$

$$= 4\pi P\left\{d - t\left(1 - \frac{1}{K}\right)\right\}.$$

If σ is the surface density of the electricity in the positive plate, $\sigma = P$, so that

$$V = 4\pi\sigma\left(d - t + \frac{t}{K}\right) \quad \text{.................(1).}$$

nce the capacity per unit area of the plate, i.e. the value of σ $V = 1$, is

$$\frac{1}{4\pi\left(d - t + \dfrac{t}{K}\right)},$$

s the same as if the plate of dielectric were replaced by a plate whose thickness was t/K. The presence of the dielectric in- the capacity of the condenser. The alteration in the capacity ot depend upon the position of the slab of dielectric between rallel plates.

us now consider the force between the plates; the force per ea

$$= \tfrac{1}{2}R\sigma,$$

R is the electric intensity at the surface of the plate; but, he surface of the plate is in contact with air, $R = 4\pi\sigma$, and thus ce per unit area on either plate

$$= 2\pi\sigma^2.$$

if the charges on the plates are given, the attraction between s not affected by the interposition of the plate of dielectric. xt, let the potentials be given; we see from equation (1) that

$$\sigma = \frac{V}{4\pi\left(d - t + \dfrac{t}{K}\right)};$$

$2\pi\sigma^2$, the force per unit area, is equal to

$$\frac{V^2}{8\pi\left(d - t + \dfrac{t}{K}\right)^2}.$$

e force between the plates when there is nothing but air between is

$$\frac{V^2}{8\pi d^2}.$$

w since K is greater than 1, $d - t + t/K$ is less than d, so that $t + t/K)^2$ is greater than $1/d^2$. Thus, when the potentials are the force between the plates is increased by the interposition dielectric.

K be very great, t/K is very small, thus $d - t + t/K$ is very nearly to $d - t$, and the effect of the interposition of the slab of di- both on the capacity and on the force between the plates is

approximately the same as if the plates had been pushed towards each other through a distance equal to the thickness of the slab, the dielectric between the plates being now supposed to be air. This result, which is approximately true whenever the specific inductive capacity of the slab is very large, is rigorously true when the slab is made of a conducting material.

Effect of the slab of dielectric on the potential energy for given charges. The potential energy is, by Art. 23, equal to

$$\tfrac{1}{2}\Sigma\,(EV),$$

and thus the energy corresponding to the charge on each unit of area of the plates is equal to

$$\tfrac{1}{2}\sigma V;$$

by equation (1) this is equal to

$$2\pi\sigma^2\left\{d - t\left(1 - \frac{1}{K}\right)\right\},$$

and it is thus when $K > 1$ less than $2\pi\sigma^2 d$, which is the value of the energy for the same charges when no slab of dielectric is interposed. The interposition of the slab thus lowers the potential energy. We can easily see why this is the case. When the charges are given the number of Faraday tubes is given: and, when the plate of dielectric is interposed, the Faraday tubes in part of their journey between the plates are in the dielectric instead of in air, and we know from Art. 73 that when the Faraday tubes are in the dielectric their energy is less than when they are in air. Since the potential energy of a system always tends to become as small as possible, there will be a tendency to drag as much as possible of the slab of dielectric between the plates of the condenser. Thus, if the slab of dielectric projected on one side beyond the plates it would be drawn in until as much of its area as possible was within the region between the plates.

Effect of the slab on the potential energy for a given difference of potential. The energy per unit area of the plates is as we have seen equal to

$$\tfrac{1}{2}\sigma V;$$

this by equation (1) is equal to

$$\frac{1}{8\pi}\frac{V^2}{\left\{d - t\left(1 - \frac{1}{K}\right)\right\}}.$$

If the potential difference is given the energy when no slab is interposed is

$$\frac{V^2}{8\pi d},$$

so that when the potential difference is kept constant the electric energy is increased by the interposition of the slab.

79. Capacity of two concentric spheres with a shell of dielectric interposed between them. If we have two concentric conducting spheres with a concentric shell of dielectric between them, and if e be the charge on the inner sphere, a the radius of this sphere and b, c the radii of the inner and outer surfaces of the dielectric shell, and d the inner radius of the outer conducting sphere, then if V be the difference of potential between the conducting spheres, and K the specific inductive capacity of the shell, we may easily prove that

$$V = e\left\{\frac{1}{a} - \frac{1}{b} + \frac{1}{K}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{1}{c} - \frac{1}{d}\right\}.$$

Thus the capacity of the system is equal to

$$\frac{1}{\frac{1}{a} - \frac{1}{d} - \left(1 - \frac{1}{K}\right)\left(\frac{1}{b} - \frac{1}{c}\right)}.$$

80. Two coaxial cylinders. As another example, we shall take the case of two coaxial cylinders with a coaxial cylindric shell of a dielectric, specific inductive capacity K, placed between them. If V be the difference of potential between the two conducting cylinders, E the charge per unit length on the inner cylinder, a the radius of this inner cylinder, b and c the radii of the inner and outer surfaces of the dielectric shell, and d the inner radius of the outer cylinder, we easily find by the aid of Art. 58 that

$$V = 2E \log\left\{\frac{b}{a} + \frac{1}{K}\log\frac{c}{b} + \log\frac{d}{c}\right\},$$

so that the capacity per unit length of this system is

$$\frac{1}{2\left\{\log\frac{b}{a} + \frac{1}{K}\log\frac{c}{b} + \log\frac{d}{c}\right\}}.$$

81. Force on a piece of dielectric placed in an electric field. If a piece of dielectric such as sulphur or glass is placed in the electric field, then, when the Faraday tubes traverse the dielectric there is, Art. 73, less energy per unit volume than when the same number of Faraday tubes pass through air. Thus, as we see in Fig. 39, the Faraday tubes tend to run through the dielectric, because by so doing the potential energy is decreased. If the dielectric is free to move, it can still further decrease the energy by moving from its original position to one where the tubes are more thickly congregated, because the more tubes which get through the dielectric the greater the decrease in the potential energy. The body will tend to move so as to make the decrease in the energy as great as possible, thus it will tend to move so as to be traversed by as great a number of Faraday tubes as possible. It will therefore be urged towards the part of the field where the Faraday tubes are densest, i.e. to the strongest parts of the field. There will thus be a force on a piece of dielectric tending to make it move from the weak to the strong parts of the field. The dielectric will not move except in a variable field where it can get more Faraday tubes by its change of position. In a uniform field such as that between two parallel infinite plates the dielectric would have no tendency to move.

The force acting upon the dielectric differs in another respect from that acting on a charged body, inasmuch as it would not be altered if the direction of the electric intensity at each point in the field were reversed without altering its magnitude.

82. Measurement of specific inductive capacity. The specific inductive capacity of a slab of dielectric can be measured in the following way, provided we have a parallel plate condenser one plate of which can be moved by means of a screw through a distance which can be accurately measured. To avoid the disturbance due to the irregular distribution of the charge near the edges of the plates (see Art. 57) care must be taken that the distance between the plates never exceeds a small fraction of the diameter of the plates. Let us call this parallel plate condenser A; to use the method described in Art. 63, first take the condenser A and before inserting the slab of dielectric adjust the other variable condenser used in that method until there is no deflection of the electrometer. If the slab of dielectric

ow inserted between the plates of A the capacity will be ased, A will no longer be balanced by the other condensers the electrometer will be deflected. The capacity of A can be ished by screwing the plates further apart, and when the plates been moved through a certain distance, the diminution in the city due to the increase in the distance between the plates alance the increase due to the insertion of the slab of dielectric; tage when this occurs will be indicated by there being again flection of the electrometer. Suppose that when the deflection e electrometer is zero before the slab is inserted, the distance een the plates of the condenser is d, while the distance after the s inserted, when the electrometer is again in equilibrium, is d'. the capacity of A in these two cases is the same. But if A area of the plate of A the capacity before the slab is inserted

$$\frac{A}{4\pi d}.$$

the thickness of the slab and K its specific inductive capacity, apacity after the insertion of the slab is (see Art. 78) equal to

$$\frac{A}{4\pi\left(d' - t + \frac{t}{K}\right)},$$

ince the capacities are equal

$$d = d' - t + \frac{t}{K},$$

at

$$d' - d = t\left\{1 - \frac{1}{K}\right\}.$$

ut $d' - d$ is the distance through which the plate has been moved, at if we know this distance and t we can determine K the specific tive capacity of the slab. It should be noticed that this method not require a knowledge of the initial or final distances between lates, but only the difference of these quantities, and this can easured with great accuracy by the screw attached to the move- plate.

his method is not applicable unless the substances whose specific tive capacities are required are exceptionally good insulators, najority of liquids do not insulate well enough to allow this

method to be used. A modification of De Sauty's method (p. 82) has been introduced by Nernst which does not demand such high insulation and which can also be used when only small quantities of the substances are available. The substance to be examined is placed in a beaker in which there are two parallel plates forming a condenser which is balanced as in De Sauty's method against an air condenser whose capacity can be graduated. The plates of this condenser as well as those of the one in which the substances are placed are short-circuited by high resistances, R_1 and R_2 respectively; these resistances are chosen so that if R_3 and R_4 are the resistances in the other arms of the system (Fig. 43 *a*), then

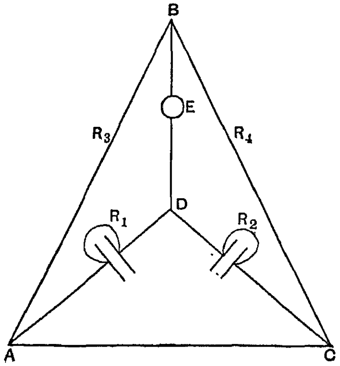


Fig. 43 *a*

$$\frac{R_1}{R_3} = \frac{R_2}{R_4}.$$

As the Wheatstone's Bridge is balanced no current will pass through BD the arm in which the electrometer is placed. Unless, however, C_1 and C_2 the capacities of the two condensers are connected by the relation

$$\frac{R_3}{C_2} = \frac{R_4}{C_1} \quad \text{..............................(1)},$$

there will be a deflection of the electrometer whenever the connection with the source of electromotive force is made or broken. Thus when there is a balance both for steady and interrupted currents we can determine C_2 the capacity of the condenser in which the substance is placed by the aid of equation (1).

To determine the specific inductive capacity the following measurements are made:

1. y_1 the capacity of the condenser C_2 when it is empty.

2. y_2 the capacity when it is filled to a definite level with the substance.

3. y_3 the capacity when it is filled to the same level with a substance whose specific inductive capacity K is known.

hen if γ_1 is the capacity of the connections, etc. which are not sed by the substance

γ_2 = the air-capacity of the parts so enclosed,

x = the specific inductive capacity of the substance,

$$y_1 = \gamma_1 + \gamma_2,$$

$$y_2 = \gamma_1 + x\gamma_2,$$

$$y_3 = \gamma_1 + K\gamma_2.$$

at x is determined by the relation

$$\frac{x-1}{K-1} = \frac{y_2 - y_1}{y_3 - y_1}.$$

CHAPTER V

ELECTRICAL IMAGES AND INVERSIOI

83. We shall now proceed to discuss some geometrical which we can find the distribution of electricity in s important cases. We shall illustrate the first method by a very simple example; that of a very small charged bo front of an infinite conducting plane maintained at po Let P, Fig. 44, be the charged body, AB the condu

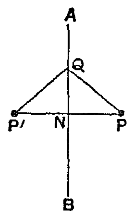


Fig. 44

Any solution of the problem must satisfy t conditions in the region to the right of th (α) it must make the potential zero over th and (β) it must make the total outward n tion taken over any closed surface enclos to $4\pi e$, where e is the charge at P, while surface does not enclose P the total norm over it must vanish. We shall now prov is only one solution which satisfies these condition there were two different solutions, which we shall c (2). Take the solution corresponding to (2) and reve of all the charges of electricity in the field, including this new solution, which we shall denote by (– 2), will to a field in which the electric intensity at any point i opposite to that due to the solution (2) at the same poin tion (– 2) corresponds to a field in which the electric zero over AB and at any point at an infinite distance fro makes the total normal induction over any closed surfa P equal to $- 4\pi e$, that is equal and opposite to the tot over the same surface due to the solution (1); and the tot over any other closed surface in the region to the right Now consider the field got by superposing the solutions (1 it will have the following properties; the potential over

zero and the total normal induction over any closed surface in the region to the right of AB will vanish. Since the normal induction vanishes over all closed surfaces in this region, there will in the field corresponding to this solution be no charge of electricity. We may regard the region as the inside of a closed surface at zero potential (bounded by the plane AB and an equipotential surface at an infinite distance): by Art. 18, however, the electric intensity must vanish throughout this region as there is no charge inside it. Thus, the electric intensity in the field corresponding to the superposition of the solutions (1) and ($-$ 2) is zero: that is, the electric intensity in the solution (1) is equal and opposite to that in ($-$ 2). But the electric intensity in ($-$ 2) is equal and opposite to that in (2). Hence the electric intensity in (1) is at all points the same as (2), in other words, the solutions give identical electric fields. Hence, if we get in any way a solution satisfying the conditions (α) and (β), it must be the only solution of the problem.

84. Let P' be a point on the prolongation of the perpendicular PN let fall from P on the plane, such that $P'N = PN$, and let a charge equal to $-e$ be placed at P'. Consider the properties, in the region to the right of AB, of the field due to the charge e at P and the charge $-e$ at P'.

The potential due to $-e$ at P' and $+e$ at P at a point Q on the plane AB is equal to

$$\frac{e}{PQ} - \frac{e}{P'Q}.$$

But since AB bisects PP' at right angles $PQ = P'Q$, thus the potential at Q vanishes. Again, any closed surface drawn in the region to the right of the plane AB does not enclose P', and thus the charge at P' is without effect upon the total induction over any such surface. The total induction over such a surface is zero or $4\pi e$ according as the closed surface does not or does include P. In the region to the right of AB the electric field due to e at P and $-e$ at P' thus satisfies the conditions (α) and (β) and therefore represents the state of the electric field. Thus the electrical effect of the electricity induced on the conducting plane AB will be the same as that of the charge $-e$ at P' at all points to the right of AB. This charge at P' is called the electrical image of the charge P in the plane.

$$\frac{e^2}{(2PN)^2} = \frac{1}{4}\frac{e^2}{PN^2}.$$

Thus the attraction on the charged body varies inversely as the square of its distance from the plane.

To find the surface density of the electricity induced on the plane AB we require the electric intensity at right angles to the plane. The electric intensity at right angles to the plane AB at a point Q on the plane due to the charge e at P is equal to

$$\frac{e}{PQ^2} \cdot \frac{PN}{PQ},$$

and acts from right to left. The electric intensity at Q due to $-e$ at P' in the same direction is

$$\frac{e}{P'Q^2} \cdot \frac{P'N}{P'Q}.$$

Hence since $PQ = P'Q$ and $PN = P'N$ the resultant normal electric intensity at Q is

$$\frac{2ePN}{PQ^3}.$$

This, by Coulomb's law, is equal to $4\pi\sigma$, if σ is the surface density of the electricity at Q, and hence

$$\sigma = \frac{e}{2\pi}\frac{PN}{PQ^3},$$

or the surface density varies inversely as the cube of the distance from P.

The total charge of electricity on the plane is $-e$, as all the tubes which start from P end on the plane.

The electrical energy is equal to $\frac{1}{2}\Sigma EV$, so that if the small body at P is a sphere of radius a, the energy in the field is equal to

$$\frac{1}{2}\frac{e^2}{a} - \frac{1}{4}\frac{e^2}{PN}.$$

The dielectric in this case is supposed to be air. The electric intensity vanishes in the region to the left of AB.

. **Electrical images for spherical conductors.** In ng the method of images to spherical tors we make great use of the follow- orem due to Apollonius. If S, Fig. 45, int on a sphere whose centre is O and a, and P and Q are two fixed points raight line passing through O, such $P \,.\, OQ = a^2$, then QS/PS is constant er S may be on the sphere.

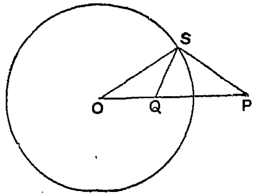


Fig. 45

nsider the triangles QOS, POS. Since

$$OQ \,.\, OP = OS^2, \quad \frac{OQ}{OS} = \frac{OS}{OP},$$

hese triangles have the angle at O common and the sides about gle proportional. They are therefore similar triangles, so that

$$\frac{QS}{OQ} = \frac{PS}{OS},$$

$$\frac{QS}{PS} = \frac{OQ}{OS} = \frac{OS}{OP}.$$

QS/PS is constant whatever may be the position of S on the .

. Now suppose that we have a spherical shell (Fig. 45) at ial zero whose centre is at O and that a small body with a charge ctricity is placed at P and that we wish to find the electric field e the sphere. There is no field inside the sphere, as the sphere quipotential surface with no charge inside it.

t $OP = f$, $OS = a$. Consider the field due to a charge e at P, at Q, where $OQ \,.\, OP = a^2$. The potential at a point S on the due to the two charges is

$$\frac{e}{PS} + \frac{e'}{QS}.$$

y Art. 85,

$$QS = PS \,.\, \frac{a}{f}.$$

the potential at $S = \left\{e + e'\,\frac{f}{a}\right\} \frac{1}{PS}.$

Hence, if $e' = -ea/f$, the potential is zero over the surface. Thus, under these circumstances the field satisfies condition (α) of Art. 83, and it obviously satisfies the condition that the total normal induction over any closed surface not enclosing the sphere is zero or $4\pi e$ according as the surface does not or does enclose P, so that, by Art. 83, this is the actual field due to the sphere and the charged body. Hence, at a point outside the sphere, the effect of the electricity induced on the sphere by the charge at P is the same as that of a charge $-ea/f$ at Q. This charge at Q is called the *electrical image of P in the sphere*. Since this charge produces the same effect as the electrification on the sphere, the total charge on the sphere must equal the charge at Q, i.e. it must be equal to $-ea/f$ (compare Art. 30). Thus of the Faraday tubes which start from P the fraction a/f fall on the sphere.

The force on P is an attraction towards the sphere and is equal to

$$\frac{a}{f}\frac{e^2}{PQ^2} = \frac{a}{f}\frac{e^2}{(OP-OQ)^2} = \frac{a}{f}\frac{e^2}{\left(f-\frac{a^2}{f}\right)^2} = \frac{e^2 fa}{(f^2-a^2)^2}.$$

We see from this result that, when the distance of P from the centre of the sphere is large compared with the radius, the force varies inversely as the *cube* of the distance from the centre of the sphere: while when P is close to the surface of the sphere the force varies inversely as the *square* of the distance from the nearest point on the surface of the sphere. When P is very near to the surface of the sphere, the problem becomes practically identical with that of a charge placed in front of a plane at potential zero. We shall leave it as an exercise for the student to deduce the solution for the plane as the limit of that of the sphere.

If the body at P is a small sphere of radius b, then since the electric energy is equal to $\frac{1}{2}\Sigma EV$, it is in this case

$$\frac{1}{2}e\left\{\frac{e}{b}-\frac{ea}{f}\frac{1}{PQ}\right\}$$

or

$$\frac{1}{2}e^2\left\{\frac{1}{b}-\frac{a}{f^2-a^2}\right\}.$$

87. To find the surface density at a point S on the surface of the sphere, we must find the electric intensity along the normal.

le of forces be resolved into the two components

$$(\alpha) \quad \frac{e}{PS^2}\frac{OS}{PS} \text{ along } OS,$$

$$(\beta) \quad \frac{e}{PS^2}\frac{OP}{PS} \text{ parallel to } PO,$$

the electric intensity at S due to the charge $-ea/f$ at Q can be ed into the components

$$(\gamma) \quad -\frac{ea}{f}\frac{1}{QS^2}\frac{OS}{QS} \text{ along } OS,$$

$$(\delta) \quad -\frac{ea}{f}\frac{1}{QS^2}\frac{OQ}{QS} \text{ parallel to } PO.$$

e the components of the resultant intensity are $\alpha + \gamma$ along the al OS, and $\beta + \delta$ parallel to PO.

ow the resultant intensity is along the normal, so that the onent $\beta + \delta$ must vanish, and the resultant intensity along the al is equal to $\alpha + \gamma$, i.e. to

$$e \,.\, OS \left\{\frac{1}{PS^3} - \frac{a}{f}\frac{1}{QS^3}\right\}$$

$$\frac{e \,.\, OS}{PS^3}\left\{1 - \frac{a}{f}\left(\frac{PS}{QS}\right)^3\right\}.$$

PS/QS is constant, the quantity inside the brackets is constant. σ is the surface density of the electrification at S, then, by omb's law,

$$4\pi\sigma = \frac{eOS}{PS^3}\left\{1 - \frac{a}{f}\left(\frac{PS}{QS}\right)^3\right\} = \frac{ea}{PS^3}\left\{1 - \frac{f^2}{a^2}\right\},$$

at the surface density of the electrification varies inversely as the of the distance from P, and is, since f is greater than a, every-e negative.

8. If the sphere is insulated instead of being at zero potential, onditions are that the potential over the sphere should be con- and that the charge on the sphere should be zero. The charge e sphere in the last case was $-ea/f$. Hence if we superpose on ast solution the field due to a quantity of electricity equal to ea/f ed at the centre of the sphere, which will give rise to a uniform

potential over the sphere, the resulting field at points outside the sphere will have the following properties: (1) the potential over the sphere is constant, (2) the total charge on the sphere is zero, (3) the total normal induction over any closed surface is equal to $4\pi e$ if the surface encloses P and is zero if it does not. Hence it is the solution in the region outside the sphere when a charge e is placed at P in front of an insulated conducting sphere. Thus, outside the insulated sphere the electric field is the same as that due to the three charges, e at P, $-ea/f$ at Q, ea/f at O. Let us consider the potential of the sphere: the charges at P and Q together produce zero potential over the sphere, so that the potential will be that due to the charge ea/f, at O; this charge produces at any point on the sphere a potential equal to e/f, so that by the presence of e at P the potential of the sphere is raised by e/f. This result was proved by a different method in Art. 29.

The force on P in this case is an attraction equal to

$$\begin{aligned} &\frac{e^2}{PQ^2}\frac{a}{f}-\frac{e^2 a}{f\,.\,f^2} \\ &= \frac{e^2 a}{f}\left\{\frac{f^2}{(f^2-a^2)^2}-\frac{1}{f^2}\right\} \\ &= \frac{e^2 a^3}{f^3}\cdot\frac{2f^2-a^2}{(f^2-a^2)^2}, \end{aligned}$$

so that in this case, when f is very large compared with a the force varies inversely as the fifth power of the distance. When the point is very close to the surface of the sphere the force is the same as if the sphere were at zero potential.

The potential energy, $\frac{1}{2}\Sigma EV$, is, if the body at P is a small sphere of radius b, equal to

$$\begin{aligned} &\frac{1}{2}e\left\{\frac{e}{b}-\frac{ea}{f\,.\,PQ}+\frac{ea}{f^2}\right\} \\ &= \frac{1}{2}e\left\{\frac{e}{b}-\frac{ea^3}{f^2(f^2-a^2)}\right\}. \end{aligned}$$

To find the surface density at S, we must superpose on the value given in Art. 87, the uniform density

$$\frac{ea}{f\,.\,4\pi a^2}.$$

Thus
$$4\pi\sigma = -\frac{ea}{PS^3}\left(\frac{f^2}{a^2}-1\right)+\frac{e}{af} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

At R the point on the sphere nearest to P,
$$PR = f - a,$$
o that the surface density at R is equal to
$$-\frac{1}{4\pi}\frac{e}{a}\left\{\frac{f+a}{(f-a)^2}-\frac{1}{f}\right\}$$
$$= -\frac{e}{4\pi f}\frac{(3f-a)}{(f-a)^2}.$$

t R' the point on the sphere most remote from P,
$$PR' = f + a,$$
nd the surface density at R' is equal to
$$\frac{e}{4\pi f}\frac{(3f+a)}{(f+a)^2}.$$

Since the total charge on the sphere is zero, the surface density f the electricity must be negative on one part of the sphere, ositive on another part. The two parts will be separated by a line n the sphere along which there is no electrification. To find the osition of this line put σ equal to zero in equation (1), we get if is a point on this line
$$PS^3 = (f^2-a^2)f = f^2\left(f-\frac{a^2}{f}\right)$$
$$= OP^2 \times PQ,$$
ence the points at which the electrification vanishes will be at a stance $(OP^2 \times PQ)^{\frac{1}{3}}$ from P.

The parts of the surface of the sphere whose distances from P e less than this value are charged with electricity of the opposite gn to that at P, the other parts of the sphere are charged with ectricity of the same sign as that at P.

89. If the sphere instead of being insulated and without charge insulated and has a charge E, we can deduce the solution by supersing on the field discussed in Art. 88 that due to a charge E unirmly distributed over the surface of the sphere; this at a point tside the sphere is the same as that due to a charge E at O. Thus

the field outside the sphere is in this case the same as that due to charges

$$E + \frac{ea}{f} \text{ at } O, \quad -\frac{ea}{f} \text{ at } Q, \quad e \text{ at } P.$$

The repulsive force acting on P is equal to

$$\left(E + \frac{ea}{f}\right)\frac{e}{f^2} - \frac{e^2 a}{f \cdot PQ^2}$$

$$= \frac{Ee}{f^2} - \frac{e^2 a^3}{f^3}\frac{(2f^2 - a^2)}{(f^2 - a^2)^2}.$$

When the point is very near the sphere we may put $f = a + x$, where x is small, and then the repulsion is approximately equal to

$$\frac{Ee}{a^2} - \frac{e^2}{4x^2}.$$

and this is negative, i.e. the force is attractive unless

$$E > e\frac{a^2}{4x^2}.$$

Thus, when the charges are given, and when P gets within a certain distance of the sphere, P will be attracted towards the sphere even though the sphere is charged with electricity of the same sign as that on P. When we recede from the sphere we reach a place where the attraction changes to repulsion, and at this point there is no force on P. Thus if P is placed at this point, it will be in equilibrium. The equilibrium will, however, be unstable, for if we displace P towards the sphere the force on it becomes attractive and so tends to bring P still nearer to the sphere, that is to increase its displacement, while if we displace P away from the sphere the force on it becomes repulsive and tends to push P still further away from the sphere, thus again increasing the displacement. This is an example of a more general theorem due to Earnshaw that no charged body (whether charged by induction or otherwise) can be in stable equilibrium in the electrostatic field under the influence of electric forces alone.

90. If the potential of the sphere is given instead of the charge, we can still use a similar method to find the field round the sphere. Thus if the potential of the sphere is V, then the field outside the sphere is the same as that due to a charge Va at O, $-ea/f$ at Q, and e at P.

1. **Sphere placed in a uniform field.** As the point P s further and further away from O the Faraday tubes due to harge at P get to be in the neighbourhood of the sphere more more nearly parallel to OP, thus when P is at a very great nce from the sphere the problems we have just considered ne in the limit problems relating to the distribution of electricity sphere placed in a uniform electric field.

ippose that, as the charged body P travels away from the e, the charge e increases in such a way that the electric intensity e centre of the sphere due to this charge remains finite and equal we have thus

$$\frac{e}{f^2} = F.$$

ow consider the problem of an insulated sphere without charge d in this uniform field. We see by Art. 88 that the electrification e sphere produces the same effect at points outside the sphere uld be produced by two charges, one equal to ea/f placed at the e O, the other equal to $-ea/f$ at Q the image of P. If we express charges in terms of F we see that they are equal respectively Faf; when f is infinite they are also infinite. Since $OQ = a^2/f$ listances between these charges diminishes indefinitely as f ases, and we see that the product of either of the charges into istance between them is equal to Fa^3 and is finite. The electrifi- n over the surface of the sphere when placed in a uniform field ices the same effect therefore as an electrical system consisting o oppositely charged bodies, placed at a very short distance , the charges on the bodies being equal in magnitude and so large the product of either of the charges into istance between them is finite. Such a system lled an electrical doublet and the product of r of the charges into the distance between is called the moment of the doublet.

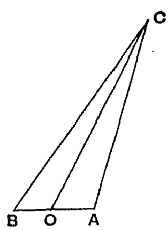


Fig. 46

2. **Electric field due to a doublet.** Let 3 be the two charged bodies, let e be the ge at A, $-e$ that at B; let O be the middle t of AB, M the moment of the doublet. Let C be a point at h the electric intensity is required, and let the angle $AOC = \theta$.

The intensity at right angles to OC is equal to

$$\frac{e}{AC^2}\sin ACO + \frac{e}{BC^2}\sin BCO$$

$$= \frac{e}{AC^3} AO \sin\theta + \frac{e}{BC^3} BO \sin\theta$$

$$= \frac{e}{OC^3} AB \sin\theta$$

$$= \frac{M\sin\theta}{OC^3},$$

approximately, since AO is very small compared with OC.

The intensity in the direction OC is equal to

$$\frac{e}{AC^2}\cos ACO - \frac{e}{BC^2}\cos BCO,$$

but we have approximately

$$AC = OC - AO\cos\theta,$$
$$BC = OC + BO\cos\theta.$$

Hence putting $\cos ACO = 1$, $\cos BCO = 1$ and using the Binomial Theorem we find that the electric intensity along OC is approximately

$$\frac{e}{OC^2}\left(1 + \frac{2AO}{OC}\cos\theta\right) - \frac{e}{OC^2}\left(1 - \frac{2BO\cos\theta}{OC}\right)$$
$$= \frac{2eAB\cos\theta}{OC^3}$$
$$= \frac{2M\cos\theta}{OC^3}.$$

93. Let us now return to the case of the sphere placed in the uniform field: the moment of the doublet which represents the effect of the electrification over the sphere is Fa^3. Hence, when the sphere is placed in a uniform field F parallel to PO, the intensity at a point C is the resultant of electric intensities, F parallel to PO, $Fa^3\sin\theta/OC^3$ at right angles to OC, and $2Fa^3\cos\theta/OC^3$ along CO; θ denotes the angle POC.

At the surface of the sphere where $OC = a$, the resultant intensity along the outward drawn normal is

$$-F\cos\theta - 2F\cos\theta,$$

or

$$-3F\cos\theta;$$

but by Coulomb's law, if σ is the surface density of the electrification on the sphere,

$$4\pi\sigma = -3F\cos\theta,$$

or

$$\sigma = -\frac{3}{4\pi}F\cos\theta.$$

Hence we see, that when an insulated conducting sphere is placed in a uniform field, the surface density at any point on the sphere is proportional to the distance of that point from a plane through the centre of the sphere at right angles to the electric intensity in the uniform field.

On account of the concentration of the Faraday tubes on the sphere the maximum intensity in the field is three times the intensity in the uniform field.

94. We have hitherto supposed the electrified body to be outside the sphere, but we can apply the same method when it is inside. Thus, if we have a charge e at a point Q inside a spherical surface maintained at zero potential, then the effect, inside the sphere, of the electricity induced on the sphere will be the same as that due to a charge $-e\,.\,a/OQ$ at P, where $OP\,.\,OQ = a^2$. The charge on the sphere is $-e$, since all the tubes which start from Q end on the sphere.

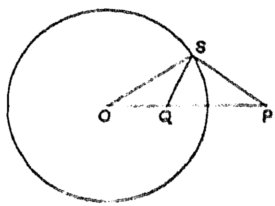


Fig. 47

If the sphere is insulated, then the charge on the inside of the sphere and the force inside are the same as when it is at potential zero; the only difference is that on the outside of the sphere there is a charge equal to e uniformly distributed over the sphere, and the field outside is the same as that due to a charge e at the centre.

Again, if there is a charge E on the sphere, the effect inside is the same as in the two previous cases, only now there is a charge $E + e$ uniformly distributed over the surface of the sphere raising its potential to $(E + e)/a$.

In all these cases the surface density of the electrification at any point on the inner surface of the sphere varies inversely as the cube of the distance of that point from P.

95. Case of two spheres intersecting at right angles and maintained at unit potential. Let the figure represent the section of the spheres, A and B being their centres, and C a point on the circle in which they intersect, CD a part of the chord common to the two circles; then, since the spheres intersect at right angles ACB is a right angle and CD is the perpendicular let fall from C on AB.

Then we have by Geometry

$$AD \,.\, AB = AC^2,$$
$$DB \,.\, AB = BC^2.$$

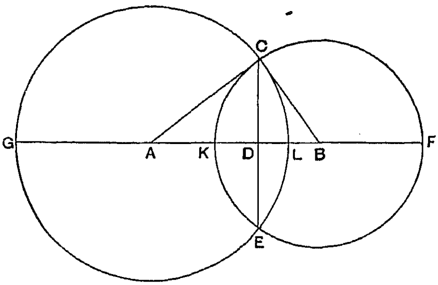


Fig. 48

Thus D and B are inverse points with regard to the sphere with centre A, and A and D are inverse points with regard to the sphere whose centre is B.

Let $AC = a$, $BC = b$, then $CD \,.\, AB = AC \,.\, BC$, so that

$$CD = \frac{ab}{\sqrt{a^2 + b^2}}.$$

Consider the effect of putting a positive charge at A numerically equal to the radius AC, a positive charge at B equal to BC, and a negative charge at D equal to CD.

The charges at A and D will together, by Art. 86, produce zero potential over the sphere with centre B. For A and D are inverse points with respect to this sphere, and the charge at D is to the charge at A as $-CD$ is to AC, i.e. as $-BC$ is to AB, so that the ratio

charges is the same as that of those on a point and its image, together produce zero potential at the sphere. Thus the value potential over the surface of this sphere is that due to the e at B, but the charge is equal to the radius of the sphere, so he potential at the surface, being equal to the charge divided e radius, is equal to unity. Thus these three charges produce potential over the sphere with centre B; we can in a similar how that they give unit potential over the sphere with centre A. two spheres then are an equipotential surface for the three es, and the electric effect of the conductor formed by the two es, when maintained at unit potential, is at a point outside the e the same as that due to the three charges.

apacity of the system. The charge on the system is equal e sum of the charges on the points inside it which produce the effect. The capacity of the system since the potential is unity ial to the charge and therefore is equal to

$$a + b - \frac{ab}{\sqrt{a^2 + b^2}}.$$

3. If b is very small compared with a, the system becomes a hemispherical boss on a large e as shown in Fig. 49. The capa-s equal to

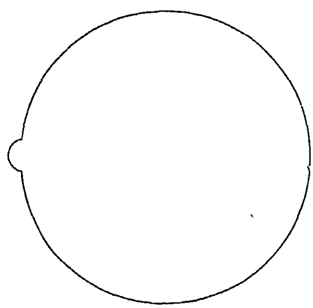


Fig. 49

$$a + b - \frac{ab}{\sqrt{a^2 + b^2}},$$

$$a\left\{1 + \frac{b}{a} - \frac{b}{a}\left(1 + \frac{b^2}{a^2}\right)^{-\frac{1}{2}}\right\};$$

as in this case b/a is very small, capacity is approximately equal

$$a\left\{1 + \frac{b}{a} - \frac{b}{a}\left(1 - \frac{1}{2}\frac{b^2}{a^2}\right)\right\}$$

$$= a\left(1 + \frac{1}{2}\frac{b^3}{a^3}\right).$$

$$\frac{1}{2}\frac{b^3}{a^3} = \frac{\text{volume of boss}}{\text{volume of big sphere}}.$$

Thus we have, since a is the capacity of the large sphere without the boss,

$$\frac{\text{increase in capacity due to boss}}{\text{capacity of sphere}} = \frac{\text{volume of boss}}{\text{volume of sphere}}.$$

97. To compare the charges on the surface of the two spheres. The charge on the spherical cap EFC (Fig. 48) is, by Coulomb's law, equal to $1/4\pi$ of the total normal induction over EFC. Now the total normal induction is the sum of the total normal inductions due to the charges at A, B, D. Since B is the centre of the cap CFE the total normal induction due to B over CFE bears the same ratio to $4\pi b$ (the total normal intensity over the whole sphere) as the area of the cap CFE does to the area of the sphere. But the area of the surface of a sphere included between two parallel planes is proportional to the distance between the planes, thus

$$\frac{\text{area of } EFC}{\text{area of sphere}} = \frac{b + BD}{2b}.$$

Hence the total normal induction over CFE due to the charge at B

$$= 2\pi(b + BD).$$

The total normal induction due to the charge A over the closed surface $CFEL$ is zero, therefore the total normal induction due to A over CFE is equal in magnitude and opposite in sign to the total normal induction over CLE, that is, it is equal to the total normal induction over CLE reckoned outwards from the side A. But CLE is a portion of a sphere of which A is the centre, therefore the induction over CLE is to $4\pi a$ (the induction over the whole sphere with centre A) as the area of CLE is to the area of the sphere, that is as $DL : 2a$. Thus the induction due to A over CFE is equal to

$$2\pi DL.$$

Next consider the total normal induction over CFE due to the charge at D. Now of the tubes starting from D as many would go to the right as to the left if it were alone in the field, so that the induction over CFE will be half that due to D over a closed surface entirely surrounding it; the latter induction is equal to 4π times the charge at D, i.e. to $-4\pi \,.\, CD$, hence the induction due to D over the surface CFE is

$$-2\pi \,.\, CD.$$

Thus the total induction over CFE due to the three charges is

$$2\pi\,(b + BD + DL - CD),$$

l the charge on CFE is therefore equal to

$$\frac{1}{2}\left(b + \frac{b^2}{\sqrt{a^2+b^2}} + a - \frac{a^2}{\sqrt{a^2+b^2}} - \frac{ab}{\sqrt{a^2+b^2}}\right) \quad \ldots\ldots(1).$$

The charge on CGE can be got by interchanging a and b in this ression, and is thus equal to

$$\frac{1}{2}\left(a + \frac{a^2}{\sqrt{a^2+b^2}} + b - \frac{b^2}{\sqrt{a^2+b^2}} - \frac{ab}{\sqrt{a^2+b^2}}\right) \quad \ldots\ldots(2).$$

98. In the case of a hemispherical boss on a large sphere, b is very all compared with a; in this case the expression (1) becomes roximately

$$\frac{1}{2}\left\{b + \frac{b^2}{a} + a - a\left(1 - \frac{1}{2}\frac{b^2}{a^2}\right) - b\right\}$$

$$= \frac{3}{4}\frac{b^2}{a}.$$

This is equal to the charge on the boss. The mean density on the s is this expression divided by $2\pi b^2$, the area of the surface of the s, and is therefore

$$\frac{3}{8\pi a}.$$

When b/a is very small the expression (2) is approximately equal a, thus the charge on the sphere is a and the mean density is got dividing a by $4\pi a^2$ the area of the sphere. Thus the mean density the sphere is

$$\frac{1}{4\pi a}.$$

nce the mean density on the boss is to the mean density on the ere as 3 : 2.

99. Since a plane may be regarded as a sphere of infinite radius, this ult applies to a hemispherical boss of any radius on a plane surface. thus applies to the case shown in Fig. 50. Since the mean density r the boss is 3/2 of that over the plane, and since the area of the s is twice the area of its base; there is three times as much eleccity on the surface occupied by the boss as there is, on the average, an area of the plane equal to the base of the boss.

100. When b is very small compared with a, the points B and D, Fig. 48, are close together, the distance between them being approximately b^2/a, which is small compared with b; the charge at B is b, that at D is

$$-\frac{ab}{\sqrt{a^2+b^2}},$$

and, when b is very small compared with a, this is approximately equal to $-b$. Thus the charges at B and D form a doublet whose moment is b^3/a. The point A is very far away and the force at B or D due to its charge is $1/a$. Thus the moment of the doublet is b^3 times this force. This as far as the sphere is concerned is exactly the case considered in Art. 93. Hence if F is the force at the boss due to the charge A alone, the surface density at a point P, Fig. 50, on the boss is $\frac{3F}{4\pi}\cos\theta$, where θ is the angle OP makes with the axis of the

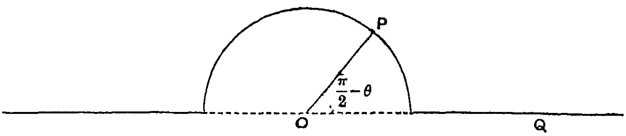


Fig. 50

doublet. Now if σ_0 is the surface density on the plane at some distance from the boss $F = 4\pi\sigma_0$. Hence, the surface density at P, a point on the boss, is equal to

$$3\sigma_0 \cos\theta,$$

where θ is the angle OP makes with the normal to the plane.

The electric intensity at Q, a point on the plane due to the doublet, is (Art. 92) equal to the moment of the doublet divided by OQ^3 and is at right angles to the plane, thus the normal electric intensity at Q is

$$F\left(1-\frac{b^3}{OQ^3}\right)$$

and σ, the surface density at Q, is given by the equation

$$\sigma = \sigma_0\left(1-\frac{b^3}{OQ^3}\right).$$

We have thus found the distribution of electricity on a charged infinite plane with a hemispherical boss on it.

. In the general case when the two spheres are of any sizes face density on the conductor can be got by calculating the electric intensity due to the three charges. We shall leave an example for the student, remarking that, since the potenthe conductor is the highest in the field, there can be no e electrification over the surface and that the electrification s along the intersection of the two spheres.

. **Effect of dielectrics.** We have hitherto only con- the case when the field due to the charge at P was disturbed presence of conductors, but by applying the principle that a which satisfies the electric conditions is the only solution, find the electric field in some simple cases when dielectrics are .

. The first case we shall consider is that of a small charged laced in front of an infinite mass of uniform dielectric bounded ane face. Let P be the charged body, AB the eparating the dielectric from air, the medium right of AB being air, that to the left a di- whose specific inductive capacity is K. From PN perpendicular to AB; produce PN to P', $PN = P'N$. Then we shall show that the field right of AB can be regarded as due to e at P harge e' at P', and that to the left of AB as due P; these charges being supposed to produce the same field as was nothing but air in the field.

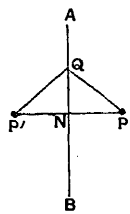


Fig. 51

he first place this field satisfies the conditions that the potenan infinite distance is zero, also that the induction over any surface surrounding P is $4\pi e$, while the induction over any surface not enclosing P is zero. This is obvious if the surface n entirely to the left or entirely to the right of AB. If it this plane it can be regarded as two surfaces, one entirely left bounded by the portion of the surface to the left and the of the plane AB intersected by the surface, the other entirely right bounded by the same portion of the plane and the part surface to the right.

only other conditions we have to satisfy are that along the

plane AB the electric intensity parallel to the surface is the same in the air as in the dielectric, and that over this plane the normal polarization is the same in the air as in the dielectric.

At a point Q in AB the electric intensity parallel to AB is in the air

$$\frac{e}{PQ^2}\frac{QN}{PQ}+\frac{e'}{P'Q^2}\frac{QN}{P'Q}.$$

This, since $PQ = P'Q$, is equal to

$$(e+e')\frac{QN}{PQ^3}.$$

The electric intensity at Q parallel to AB in the dielectric is

$$e''\frac{QN}{PQ^3};$$

this is equal to that in air if

$$e+e'=e'' \quad \text{..........................(1)}.$$

Again, the polarization at Q at right angles to AB reckoned from right to left is in air

$$\frac{1}{4\pi}(e-e')\frac{PN}{PQ^3},$$

and that in the dielectric is

$$\frac{K}{4\pi}e''\frac{PN}{PQ^3};$$

these are equal if

$$e-e'=Ke'' \quad \text{..........................(2)}.$$

Hence both the boundary conditions are satisfied if e' and e'' satisfy (1) and (2), i.e. if

$$e''=\frac{2}{1+K}e,$$

$$e'=-\frac{(K-1)}{K+1}e.$$

The attraction of P towards the plane is equal to that between e and e' and is thus

$$-\frac{ee'}{(2PN)^2}=\frac{K-1}{K+1}\frac{e^2}{4PN^2}.$$

If K is infinite this equals

$$\frac{e^2}{4PN^2};$$

which is the same as when the dielectric to the left of AB is replaced by a conductor.

the attraction when P is placed in front of a conducting plate. ide the mass of dielectric the tubes are straight and would if longed all pass through P; the effect of the dielectric is, while affecting the direction of the electric intensity, to reduce its gnitude to $2/(1 + K)$ of its value in air when the dielectric is reved. The lines of force when $K = 1{\cdot}7$ are shown in Fig. 52.

104. Case of a dielectric sphere placed in a uniform ld. We have seen that, when a conducting sphere is placed in

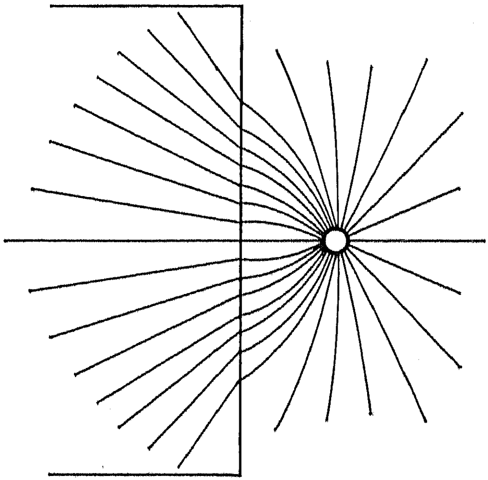


Fig. 52

niform field, the effect of the electricity induced on the surface the sphere can be represented at points outside the sphere by a ıblet (see Art. 92) placed at the centre of the sphere. Since we have n that the effects of a dielectric are similar in kind though different degree to those due to a conductor, we are led to try if the disbance produced by the presence of the sphere cannot be reprented at a point outside the sphere by a doublet placed at its centre. th regard to the field inside the sphere we have as a guide the ult obtained in the last article, that in the case when the radius the sphere is infinitely large the field inside the dielectric is not ered in direction but only in magnitude by the dielectric.

We therefore try if we can satisfy the conditions which must hold when a sphere is placed in a uniform electric field by supposing the field inside the sphere to be uniform.

Let the uniform field before the insertion of the sphere be one where the electric intensity is horizontal and equal to H.

After the insertion of the sphere let the field outside consist of this uniform field plus the field due to a doublet whose moment is M placed at the centre of the sphere, the dielectric being removed.

Inside the sphere let the intensity be horizontal and equal to H'.

We shall see that it is possible to satisfy the conditions of the problem by a proper choice of M and H'.

The field at P due to the doublet is, by Art. 92, equivalent to an intensity $\frac{2M}{OP^3}\cos\theta$ along OP, and an intensity $\frac{M}{OP^3}\sin\theta$ at right angles to it, where θ is the angle OP makes with the direction of the uniform electric intensity. Thus at a point Q just outside the sphere the intensity tangential to the sphere is equal to

$$H\sin\theta - \frac{M}{a^3}\sin\theta,$$

where a is the radius of the sphere.

The intensity in the same direction at a point close to Q but just inside the sphere is

$$H'\sin\theta.$$

The normal intensity at Q outside the sphere is

$$H\cos\theta + \frac{2M}{a^3}\cos\theta,$$

and at a point just inside the sphere it is $H'\cos\theta$.

The first boundary condition is that the tangential intensity at the surface of the sphere must be the same in the air as in the dielectric; this will be true if

$$H\sin\theta - \frac{M}{a^3}\sin\theta = H'\sin\theta,$$

or

$$H - \frac{M}{a^3} = H' \quad \text{..........................(1).}$$

The second boundary condition is that the normal polarization the surface of the sphere must be the same in the air as in the electric, thus

$$\frac{1}{4\pi}\left\{H\cos\theta + \frac{2M}{a^3}\cos\theta\right\} = \frac{K}{4\pi}H'\cos\theta,$$

$$H + \frac{2M}{a^3} = KH' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Equations (1) and (2) will be satisfied, if

$$H' = \frac{3H}{K+2},$$

d if

$$M = \frac{H(K-1)}{K+2}a^3.$$

Thus, since, if H' and M have these values the conditions are tisfied, this will be the solution of the problem. We see that the tensity inside the sphere is $3/(K+2)$ of that in the original field, that the intensity of the field is less inside the sphere than outside; the other hand the number of Faraday tubes which pass through it area inside the sphere is $3K/(K+2)$ times the number passing rough unit area in the original uniform field. When K is very great $K/(K+2)$ is approximately equal to 3, so that the Faraday tubes this case will be 3 times as dense inside the sphere as they are a great distance away from it. This illustrates the crowding of the raday tubes to the sphere.

The diagram of the lines of force for this case was given in Fig. 41.

Method of Inversion.

105. This is a method by which, when we have obtained the ution of any problem in electrostatics, we can by a geometrical ocess obtain the solution of another.

Definition of inverse points. If O is a fixed point, P a riable one, and if we take P' on OP, so that

$$OP \,.\, OP' = k^2,$$

ere k is a constant, then P' is defined to be the inverse point of P th regard to O, while O is called the centre of inversion, and k e radius of inversion.

If the point P moves about so as to trace out a surface, then P' will trace out another surface which is called the surface inverse to that traced out by P.

We shall now proceed to prove some geometrical propositions about inversion.

106. The inverse surface of a sphere is another sphere. Let O be the centre of inversion, P a point on the sphere to be inverted, C the centre of this sphere. Let the chord OP cut the sphere again in P', let Q be the point inverse to P, Q' the point inverse to P', R the radius of the sphere to be inverted, then

$$OP \,.\, OQ = k^2.$$

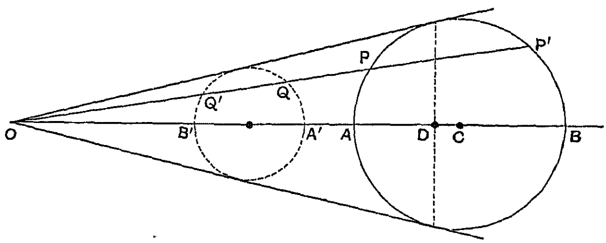


Fig. 53

But $$OP \,.\, OP' = OC^2 - R^2,$$

and thus $$OQ = \frac{k^2}{OC^2 - R^2} OP';$$

similarly $$OQ' = \frac{k^2}{OC^2 - R^2} OP,$$

and $$OQ \,.\, OQ' = \frac{k^4}{(OC^2 - R^2)^2} OP \,.\, OP'$$

$$= \frac{k^4}{OC^2 - R^2}.$$

Thus OQ bears a constant ratio to OP'; hence the locus of Q is similar to the locus of P', and is therefore a sphere. Thus a sphere inverts into a sphere. If

$$k^2 = OC^2 - R^2$$

the sphere inverts into itself.

To find the centre of the inverse sphere, let the diameter OC cut he sphere to be inverted in A and B. Let A', B' be the points inverse o A and B respectively and O' the centre of the inverted sphere; hen

$$OO' = \frac{1}{2}(OA' + OB')$$
$$= \frac{1}{2}\left(\frac{k^2}{OC - R} + \frac{k^2}{OC + R}\right)$$
$$= k^2 \cdot \frac{OC}{OC^2 - R^2}.$$

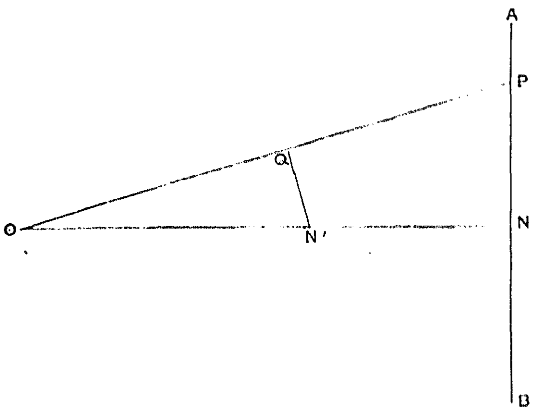


Fig. 54.

If D is the point where the chord of contact of tangents from O o the sphere cuts OC, then

$$OD = \frac{OC^2 - R^2}{OC}.$$

Hence D inverts into the centre of the sphere.

The radius of the inverse sphere

$$= \tfrac{1}{2}(OA' - OB')$$
$$= k^2 \cdot \frac{R}{OC^2 - R^2}.$$

107. Since a plane is a particular case of a sphere a plane will nvert into a sphere; this can be proved independently in the following way:

Let AB be the plane to be inverted, P a point on that plane, N the foot of the perpendicular let fall from O on the plane and Q and N' the points inverse to P and N respectively. Then since

$$OQ \,.\, OP = ON' \,.\, ON,$$

$$\frac{OQ}{ON'} = \frac{ON}{OP};$$

thus the two triangles QON', PON have the angle at O common and the sides about this angle proportional, they are therefore similar, and the angle OQN' is equal to the angle ONP. Hence OQN' is a right angle and therefore the locus of Q is a sphere on ON' as diameter.

108. Let O be the centre of inversion, PQ two points and $P'Q'$ the corresponding inverse points.

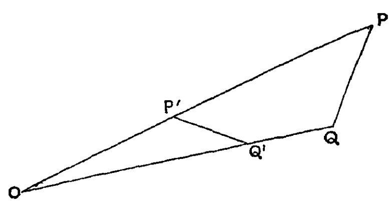


Fig. 55

Then $$\frac{OP'}{OQ'} = \frac{OQ}{OP};$$

thus the triangles POQ, $Q'OP'$ are similar, so that

$$\frac{PQ}{OP} = \frac{P'Q'}{OQ'}.$$

If we have a charge e at Q, and a charge e' at Q', then if V_P is the potential at P due to the charge at Q, and $V'_{P'}$ the potential at P' due to the charge at Q',

$$V_P : V'_{P'} = \frac{e}{PQ} : \frac{e'}{P'Q'} = \frac{e}{OP} : \frac{e'}{OQ'}.$$

Take $$e : e' = OQ : k \quad \text{..........................(1)},$$

then $$V'_{P'} = V_P \frac{k}{OP'}.$$

If we have any number of charges at different points and take inverse of these points and place there charges given by the ression (1), then, if $\overline{V}_P$ be the potential at a point P due to the ginal assemblage of charges, $\overline{V}_{P'}$ the potential at P' (the point erse to P) due to the charges on the inverted system,

$$\overline{V}_{P'} = \overline{V}_P \frac{k}{OP'}.$$

s, if the original assemblage of charges produces a constant ential V over a surface S, the inverted system will produce a ential $\frac{Vk}{OP'}$ at a point P' on the inverse of S. Hence, if we add he inverted system a charge $-kV$ at the centre of inversion, potential over the inverse of S will be zero.

If the charges on the original system are distributed over a ace instead of being concentrated at points the charges on the erted system will also be distributed over a surface. Let σ be the ace density at Q, a place on the original system, σ' the surface sity at Q', the corresponding point on the inverted system, α a ll area at Q, α' the area into which it inverts; then by (1)

$$\sigma\alpha : \sigma'\alpha' = OQ : k$$

, since α and α' are similar figures,

$$\alpha : \alpha' = OQ^2 : OQ'^2.$$

ce
$$\sigma : \sigma' = OQ'^2 : kOQ$$

thus
$$\sigma' = \sigma \frac{kOQ}{OQ'^2} = \sigma \frac{k^3}{OQ'^3} \quad \text{.....................(2).}$$

This expression gives the surface density of the inverted figure erms of that at the corresponding point of the original figure.

109. As an example of the use of the method of inversion let us ert the system consisting of a sphere with a uniform distribution lectricity over it, the surface density being $V/4\pi a$; where a is the ius of the sphere. We know in this case that the potential is stant over the sphere and equal to V. Take the centre of inversion side the sphere and choose the radius of inversion so that the ere inverts into itself. Then, if to the inverted system we add a rge $-kV$ at the centre of inversion the inverse sphere will be at

potential zero. By equation (2) σ' the surface density in the inverted system at Q' is given by the equation

$$\sigma' = \frac{V}{4\pi a} \cdot \frac{k^3}{OQ'^3}.$$

If we put $e = -kV$, this equals

$$\frac{-e}{4\pi a} \cdot \frac{k^2}{OQ'^3} = \frac{-e \,.\, (OC^2 - a^2)}{4\pi a \,.\, OQ'^3},$$

where C is the centre of the sphere.

Thus a charge e at O induces on the sphere at zero potential a distribution of electricity such that the surface density varies inversely as the cube of the distance from O. In this way we get by inversion the solution of the problem which we solved in Art. 87 by the method of images.

110. As an example illustrating the uses of the method of inversion as well as that of images, let us consider the solution, by the method of images, of a charged body placed between two infinite conducting planes maintained at potential zero.

Let P be the charged point, AB and CD the two planes at potential zero, e the charge at P. Then if we place a charge $-e$ at P' where P' is the image of P in AB the potential over AB will be zero, it will not however be zero over CD; to make the potential over CD zero we must place a charge $-e$ at Q, the image of P in CD, and a charge e at Q_1, the image of P' in CD. These two charges will however disturb the potential of AB; to restore zero potential to AB we must introduce a charge $+e$ at P_1, the image of Q in AB, and a charge $-e$ at P'', the image of Q_1 in AB. The charges at P_1 and P'' will disturb the potential over the plane CD; to restore it to zero we must place a charge $-e$ at Q', the image of P_1 in CD, and a charge $+e$ at Q_2, the image of P'' in CD, and so on; we get in this way two infinite series of images to the right of AB and to the left of CD.

The images to the right of AB are (1) charges $-e$, at P', P'', $P'''\ldots$; and (2) charges $+e$, at P_1, P_2, $P_3\ldots$.

Now P'' is the image of Q_1 in AB, which is the image of P' in CD and hence

$$FP'' = FQ_1 = FE + EP' = 2FE + FP';$$

distances

thus $FP'' - FP' = P'P'' = 2FE = 2c$, if c is the distance between the plates.

Similarly $P'P'' = P''P''' = \ldots = 2c$ and we can show in a similar way that $PP_1 = P_1P_2 = P_2P_3 = \ldots = 2c$. Thus on the right of AB we have an infinite series of charges equal to $-e$ at the distance $2c$ apart, beginning at P' the image of P in AB, and a series of positive images at the same distance $2c$ apart, beginning at P_1, a point distant $2c$ from P.

Similarly to the left of CD we have an infinite series of images with the charge $-e$ at the distance $2c$ apart, beginning at Q, the image of P in CD, and an infinite series of images each with the charge $+e$, at points at a distance $2c$ apart, beginning at Q_1, a point distant $2c$ from P.

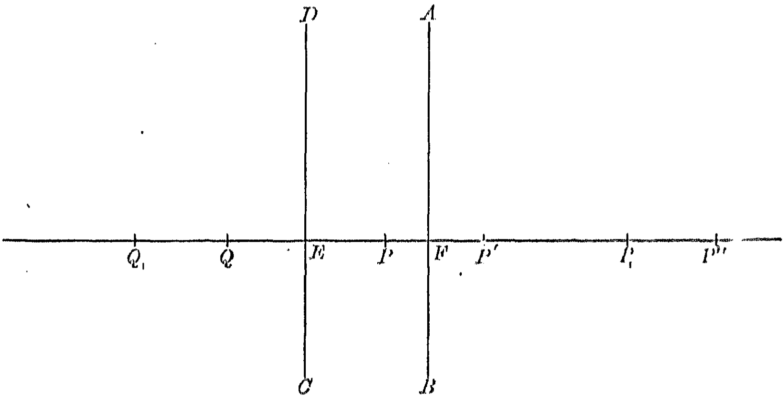


Fig. 56

Now invert this system with respect to P. The two planes invert into two spheres touching each other at P, and maintained at a potential $-e/k$, the images to the right of AB invert into a series of charged points inside the sphere to the right of P and the images to the left of CD invert into a system of charged points inside the sphere to the left of P.

The system of charged points inside the spheres will produce a constant potential $-e/k$ over the surface of the spheres, and therefore at a point outside the spheres the electric field due to the two spheres in contact will be the same as that due to the system of the electrified points.

If a, b are the radii of the spheres into which the planes AB, CD invert, and if $PF = d$, then

$$2a = \frac{k^2}{d} \qquad c = \frac{1}{2}\left\{\frac{k^2}{a} + \frac{k^2}{b}\right\}.$$

$$2b = \frac{k^2}{c-d}$$

Consider now the series of images to the right of AB. The series of positive charges at the distance $2c$ apart invert into a series of charges inside the sphere, whose radius is a, of magnitudes

$$\frac{ek}{2c},\ \frac{ek}{4c},\ \frac{ek}{6c},\ \dots,$$

since

$$\frac{\text{charge at inverted point}}{\text{charge at original point}}$$

$$= \frac{k}{\text{distance of original point from centre of inversion}}.$$

The series of negative images at the distance $2c$ apart invert into a series of negative charges

$$-\frac{ek}{2d},\quad -\frac{ek}{2c+2d},\quad -\frac{ek}{4c+2d},\ \dots.$$

Similarly, inside the sphere into which the plane CD inverts, we have a series of positive charges

$$\frac{ek}{2c},\ \frac{ek}{4c},\ \frac{ek}{6c},\ \dots,$$

and a series of negative ones

$$-\frac{ek}{2(c-d)},\quad -\frac{ek}{4c-2d},\quad -\frac{ek}{6c-2d},\ \dots.$$

Thus E_1, the sum of the charges on the points inside the first sphere, is given by the equation

$$E_1 = ek\left\{\left(\frac{1}{2c} + \frac{1}{4c} + \frac{1}{6c} + \dots\right)\right.$$

$$\left.- \left(\frac{1}{2d} + \frac{1}{2c+2d} + \frac{1}{4c+2d} + \dots\right)\right\} \dots (1),$$

le E_2, the sum of the charges inside the second sphere, is given the equation

$$E_2 = ek\left\{\left(\frac{1}{2c} + \frac{1}{4c} + \frac{1}{6c} + \dots\right) - \left(\frac{1}{2c-2d} + \frac{1}{4c-2d} + \dots\right)\right\}\dots(2).$$

Rearranging the terms, we may write

$$E_1 = -\frac{1}{2}ek\left\{\frac{1}{d} - \frac{d}{c(c+d)} - \frac{d}{2c(2c+d)} - \frac{d}{3c(3c+d)} - \dots\right\},$$

$$E_2 = -\frac{1}{2}ek\frac{d}{c}\left\{\frac{1}{c-d} + \frac{1}{2(2c-d)} + \frac{1}{3(3c-d)} + \dots\right\}.$$

Expanding the expressions for E_1 and E_2 in powers of d/c we get

$$E_1 = -\frac{1}{2}ek\left(\frac{1}{d} - \frac{d}{c^2}S_2 + \frac{d^2}{c^3}S_3 - \frac{d^3}{c^4}S_4 + \dots\right)\dots\dots\dots(3),$$

$$E_2 = -\frac{1}{2}ek\frac{d}{c^2}\left(S_2 + \frac{d}{c}S_3 + \frac{d^2}{c^2}S_4 + \frac{d^3}{c^3}S_5 + \dots\right),$$

ere $$S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots.$$

The values of S_n are given in De Morgan's *Differential and Integral culus*, p. 554,

$$S_2 = \frac{\pi^2}{6} = 1{\cdot}645, \qquad S_5 = 1{\cdot}037,$$

$$S_3 = 1{\cdot}202, \qquad S_6 = \frac{\pi^6}{945} = 1{\cdot}017,$$

$$S_4 = \frac{\pi^4}{90} = 1{\cdot}082, \qquad S_7 = 1{\cdot}008.$$

Since E_1 can be got from E_2 by writing $c - d$ for d, we get

$$E_1 = -\frac{1}{2}ek\frac{(c-d)}{c^2}\left(S_2 + \frac{c-d}{c}S_3 + \frac{(c-d)^2}{c^2}S_4 + \dots\right)\dots(4).$$

Now, the total charge spread over the surface of the first sphere equal to the sum of the charges at the points inside the sphere these produce the same effect at external points as the electrification over the surface of the sphere: thus, E_1 will be the charge on

the first sphere, E_2 that on the second. If V is the potential of the spheres

$$V = \frac{e}{k},$$

hence

$$E_1 = Va\left(1 - \frac{b}{a+b}\,\frac{b}{a+2b} - \frac{b}{2(a+b)}\,\frac{b}{2a+3b} - \frac{b}{3(a+b)}\,\frac{b}{3a+4b} - \dots\right) \dots(5),$$

$$E_2 = Vb\left(1 - \frac{a}{a+b}\,\frac{a}{2a+b} - \frac{a}{2(a+b)}\,\frac{a}{3a+2b} - \frac{a}{3(a+b)}\,\frac{a}{4a+3b} - \dots\right) \dots(6),$$

$$E_1 = Va\left\{1 - \left(\frac{b}{a+b}\right)^2 S_2 + \left(\frac{b}{a+b}\right)^3 S_3 - \dots\right\} \quad\dots\dots\dots\dots(7),$$

and also

$$E_1 = V\frac{ba^2}{(a+b)^2}\left\{S_2 + \left(\frac{a}{a+b}\right) S_3 + \left(\frac{a}{a+b}\right)^2 S_4 + \dots\right\} \dots(8).$$

The value of E_2 can be got by interchanging a and b in the expressions (7) and (8).

Let us now consider some special cases. Take first the case when $a = b$, then from equation (5) we have

$$\begin{aligned} E_1 &= Va\left\{1 - \frac{1}{2}\,\frac{1}{3} - \frac{1}{4}\,\frac{1}{5} - \frac{1}{6}\,\frac{1}{7} - \dots\right\} \\ &= Va\left\{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots\right\} \\ &= Va \log 2, \end{aligned}$$

the logarithm being the Napierian logarithm.

Since $\log 2 = \cdot 693$

$$E_1 = \cdot 693\, Va.$$

The charge on the second sphere is also E_1; thus the total charge on the two spheres is

$$1\cdot 386\, Va.$$

When $V = 1$ the charge on the two spheres is equal to the capacity of the system; hence the capacity of two equal spheres in contact is $2a \log 2$ or $1\cdot 386\, a$.

If the spheres had been an infinite distance apart, the capacity the two would have been $2a$; if there had only been one sphere the pacity would have been a.

We can find from this the work done on an uncharged sphere ıen it moves under the attraction of a charged sphere of equal dius from an infinite distance into contact with the charged sphere. t a be the radius of each sphere and e the charge on the charged here; then, when the spheres are at an infinite distance apart, the tential energy is $e^2/2a$ and when the spheres are in contact the tential energy is $e^2/2 \times 1{\cdot}386a$. Hence the work done by the electric ld while the uncharged sphere falls from an infinite distance into ıtact with the charged sphere is

$$\frac{1}{2}\frac{e^2}{a}\left\{1 - \frac{1}{1{\cdot}386}\right\} = {\cdot}14\frac{e^2}{a}.$$

If one sphere has a charge E, the other the charge e, then, when ey are at an infinite distance apart, the potential energy is

$$\frac{1}{2a}\{E^2 + e^2\}.$$

When the spheres are in contact the potential energy is

$$\frac{1}{2 \times 1{\cdot}386a}\{E + e\}^2.$$

Hence the potential energy is greater in the second case than in ə first by

$$-\frac{1}{a}\left\{{\cdot}14E^2 - \frac{1}{1{\cdot}386}Ee + {\cdot}14e^2\right\} \quad \ldots\ldots\ldots\ldots(9).$$

If $E = e$, this is equal to

$$+\frac{{\cdot}44e^2}{a}.$$

This is the work required to push the spheres together against ə repulsions exerted by their like charges.

The expression (9) vanishes when E/e is approximately 5 or 1/5; this case the potential energy is the same when the spheres are in ntact as when they are an infinite distance apart; thus no work is ent or gained in bringing them together. The attraction due to the duced electrification on the average balances the repulsion due to ə like charges.

by (8) we have

$$E_1 = \frac{Vba^2}{(a+b)^2}\left(S_2 + \frac{a}{a+b} S_3 + \ldots\right),$$

or approximately, when b/a is large,

$$E_1 = \frac{Va^2}{b} S_2$$

$$= V \frac{a^2}{b} \frac{\pi^2}{6}$$

$$= 1{\cdot}645 \frac{Va^2}{b}.$$

Interchanging a and b in (7) we get

$$E_2 = Vb\left\{1 - \left(\frac{a}{a+b}\right)^2 S_2 + \left(\frac{a}{a+b}\right)^3 S_3 - \ldots\right\}$$

or approximately, when b/a is large,

$$E_2 = V\left(b - \frac{a^2}{b} S_2\right)$$

$$= V\left(b - \frac{\pi^2}{6}\frac{a^2}{b}\right) = V\left(b - 1{\cdot}645\frac{a^2}{b}\right).$$

The mean surface density over the small sphere is

$$\frac{E_1}{4\pi a^2} = \frac{V}{4\pi b} \cdot \frac{\pi^2}{6} = \frac{V}{4\pi b} 1{\cdot}645.$$

The mean surface density over the large sphere is approximately

$$\frac{E_2}{4\pi b^2} = \frac{V}{4\pi b}$$

and hence the mean surface density on the small sphere is $\pi^2/6$ or 1·645 times that on the large sphere. We saw in Art. 98 that, when a small hemisphere was placed on a large sphere, the mean density on the hemisphere was 1·5 times that on the sphere.

Since a plane may be regarded as a sphere of infinite radius, we see that if a sphere of any size is placed on a conducting plane the mean surface density of the electricity on the sphere is $\pi^2/6$ of that on the plane.

have

$$E_1 + E_2 = Vb\left\{1 + \frac{2a^3}{(a+b)^3}S_3 + \dots\right\}$$

$$= Vb\left\{1 + 2{\cdot}404\,\frac{a^3}{b^3}\right\} \text{ approximately.}$$

us, the capacity of the system of two spheres is approximately

$$b\left\{1 + 2{\cdot}404\frac{a^3}{b^3}\right\}.$$

have thus

$$\frac{\text{rease of capacity due to small sphere}}{\text{Capacity of large sphere}}$$

$$= 2{\cdot}404\,\frac{\text{volume of small sphere}}{\text{volume of large sphere}}.$$

us in this case, as in that discussed in Art. 96, the increase of ty due to the small body is proportional to the volume of the body.

om this result we can deduce the work done on a small un-d sphere of radius a when it moves from an infinite distance a large sphere of radius b with a charge E.

r, when they are at an infinite distance apart, the potential is equal to

$$\frac{1}{2}\frac{E^2}{b};$$

the spheres are in contact the potential energy is

$$\frac{1}{2}\frac{E^2}{b\left\{1 + 2{\cdot}404\frac{a^3}{b^3}\right\}}.$$

e work done on the small sphere by the electrical forces is the nce between these expressions, or approximately,

$$E^2\,1{\cdot}202\frac{a^3}{b^4}.$$

CHAPTER VI

MAGNETISM

111. A mineral called 'lodestone' or magnetic oxide of iron, which is a compound of iron and oxygen, is often found in a state in which it possesses the power of attracting small pieces of iron such as iron filings; if the lodestone is dipped into a mass of iron filings and then withdrawn, some of the iron filings will cling to the lodestone, collecting in tufts over its surface. The behaviour of the lodestone is thus in some respects analogous to that of the rubbed sealing-wax in the experiment described in Art. 1. There are however many well-marked differences between the two cases; thus the rubbed sealing-wax attracts all light bodies indifferently, while the lodestone does not show any appreciable attraction for anything except iron and, to a much smaller extent, nickel and cobalt.

If a long steel needle is stroked with a piece of lodestone, it will acquire the power possessed by the lodestone of attracting iron filings; in this case the iron filings will congregate chiefly at two places, one at each end of the needle, which are called the poles of the needle.

The piece of lodestone and the needle are said to be magnetized; the attraction of the iron filings is an example of a large class of phenomena known as magnetic. Bodies which exhibit the properties of the lodestone or the needle are called *magnets*, and the region around them is called the *magnetic field*.

The property of the lodestone was known to the ancients, and is frequently referred to by Pliny and Lucretius. The science of Magnetism is indeed one of the oldest of the sciences and attained considerable development long before the closely allied science of Electricity; this was chiefly due to Gilbert of Colchester, who in his work *de Magnete* published in 1600 laid down in an admirable manner the cardinal principles of the science.

112. Forces between Magnets. If we take a needle which has been stroked by a lodestone and suspend it by a thread attached

entre it will set itself so as to point in a direction which is not
ır from north and south. Let us call the end of the needle
points to the north, the north end, that which points to the
the south end, and let us when the needle is suspended mark
l which is to the north; let us take another needle, rub it with
estone, suspend it by its centre and again mark the end which
the north. Now bring the needles together; they will be found
t forces on each other, and the two ends of a needle will be
to possess sharply contrasted properties. Thus if we place
gnets so that the two marked ends are close together while
nmarked ends are at a much greater distance apart, the marked
ill be repelled from each other; again, if we place the magnets
the two unmarked ends are close together while the marked
e at a much greater distance apart, the unmarked ends will be
to be repelled from each other; while if we place the two
ts so that the marked end of one is close to the unmarked end
other, while the other ends are much further apart, the two
hich are near each other will be found to be attracted towards
her. We see then that poles of the same kind are repelled from
ther, while poles of opposite kinds are attracted towards each
Thus the two ends of a magnet possess properties analogous
se shown by the two kinds of electricity.

3. We shall find it conduces to brevity in the statement of
ws of magnetism to introduce the term *charge of magnetism*,
express the property possessed by the ends of the needles in
eceding experiment by saying that they are charged with
tism, one end of the needle being charged with positive
tism, the other end with negative. We regard the end of the
which points to the north as having a charge of positive
tism, the end which points to the south as having a charge
ative magnetism. It will be seen from the preceding experi-
that two charges of magnetism are repelled from or attracted
ds each other according as the two charges are of the same or
te signs. It must be distinctly understood that this method
arding the magnets and the magnetic field is only introduced
rding a convenient method of *describing* briefly the phenomena
t field and not as having any significance with respect to the

constitution of magnets or the mechanism by which the forces are produced: we saw for example that the same terminology afforded a convenient method of describing the electric field, though we ascribe the action in that field to effects taking place in the dielectric between the charged bodies rather than in the charged bodies themselves.

114. Unit Charge of Magnetism, often called *pole of unit strength.* Take two very long, thin, uniformly magnetized needles, equal to each other in every respect (we can test the equality of their magnetic properties by observing the forces they exert on a third magnet), let A be one end of one of the magnets, B the like end of the other magnet, place A and B at unit distance apart in air, the other ends of the magnets being so far away that they exert no appreciable effect in the region about A and B: then each of the ends A and B is said to have a unit charge of magnetism or to be a pole of unit strength when A is repelled from B with the unit force. If the units of length, mass and time are respectively the centimetre, gramme and second the force between the unit poles is one dyne.

A charge of magnetism equal to 2, or a pole of strength 2, is one which would be repelled with the force of two dynes from unit charge placed at unit distance in air.

If m and m' are the charges on two ends of two magnets (or the strengths of the two poles), the distance between the charges being the unit distance, the repulsion between the charges is mm' dynes. If the charges are of opposite signs mm' is negative: we interpret a negative repulsion to mean an attraction.

115. Coulomb by means of the torsion balance succeeded in proving that the repulsion between like charges of magnetism varies inversely as the square of the distance between them. We shall discuss in Art. 132 a more delicate and convenient method of proving this result.

Since the forces between charges of magnetism obey the same laws as those between electric charges we can apply to the magnetic field the theorems which we proved in Chap. II. for forces varying inversely as the square of the distance.

The Magnetic Force at any point is the force which ct on unit charge if placed at this point, the introduction harge being supposed not to influence the magnets in the

Magnetic Potential. The magnetic potential at a is the work which would be done on unit charge by the forces if it were taken from P to an infinite distance. We e as in Art. 17 that the magnetic potential due to a charge istance r from the charge is equal to m/r.

The total charge of Magnetism on any magnet This is proved by the fact that if a magnet is placed form field the resultant force upon it vanishes. The earth a magnet and produces a magnetic field which may be l as uniform over a space enclosed by the room in which riments are made. To show the absence of any horizontal t force on a magnet, we may mount the magnet on a piece and let this float on a basin of water, then though the will set so as to point in a definite direction, there will be ency for the magnet to move towards one side of the basin. a couple acting on the magnet tending to twist it so that gnet sets in the direction of the magnetic force in the field, e is no resultant horizontal force on the magnet. The absence vertical force is shown by the fact that the process of magn has no influence upon the weight of a body. Either of sults shows that the total charge on the body is zero. For m_2, m_3, &c. be the magnetic charges on the body, F the l magnetic force, then the total force acting on the body in ction of F is

$$\Sigma Fm.$$

nce the field is uniform, is equal to $F\Sigma m$.
his vanishes $\Sigma m = 0$, i.e. the total charge on the body is lence on any magnet the positive charge is always equal to ative one.

n considering electric phenomena we saw that it was im- to get a charge of positive electricity without at the same etting an equal charge of negative electricity. It is also ble to get a charge of positive magnetism without at the

same time getting an equal charge of negative magnetism; but whereas in the electrical case all the positive electricity might be on one body and all the negative on another, in the magnetic case if a charge of positive magnetism appears on a body an equal charge of negative magnetism must appear on the *same* body. This difference between the two cases would disappear if we regarded the *dielectric* in the electrical case as analogous to the magnets; the various charged bodies in the electrical field being regarded as portions of the surface of the dielectric.

119. Poles of a Magnet. In the case of very long and thin uniformly magnetized pieces of iron and steel we approximate to a state of things in which the magnetic charges can be regarded as concentrated at the ends of the magnet, which are then called its poles; the positive magnetism being concentrated at the end which points to the north, which is called the positive pole, the negative charge at the other end, called the negative pole.

In general however the magnetic charges are not localized to such an extent as in the previous case, they exist more or less over the whole surface of the magnet; to meet these cases we require a more extended definition of 'the pole of a magnet.'

Suppose the magnet placed in a uniform field, then the forces acting on the positive charges will be a series of parallel forces all acting in the same direction, these by statics may be replaced by a single force acting at a point P called the centre of parallel forces for this system of forces. This point P is called the positive pole of the magnet. Similarly the forces acting on the negative charges may be replaced by a single force acting at a point Q. This point Q is then called the negative pole of the magnet. The resultant force acting at P is by statics the same as if the whole positive charge were concentrated at P; this resultant is equal and opposite to that acting at Q.

120. Axis of a Magnet. The axis of a magnet is the line joining its poles, the line being drawn from the negative to the positive pole.

121. Magnetic Moment of a Magnet is the product of the charge of positive magnetism multiplied by the distance between the poles. It is thus equal to the couple acting on the magnet when placed in a uniform magnetic field where the intensity of the magnetic

ce is unity, the axis of the magnet being at right angles to the
ection of the magnetic force in the uniform field.

122. The Intensity of Magnetization is the magnetic
ment of a magnet per unit volume. It is to be regarded as having
ection as well as magnitude, its direction being that of the axis
the magnet.

123. Magnetic Potential due to a Small Magnet. Let
and B, Fig. 57, represent the poles of a small magnet, m the charge
magnetism at B, $-m$ that at A.
t O be the middle point of AB.
nsider the magnetic potential at
due to the magnet AB. The mag-
ic potential at P due to m at B is
, that due to $-m$ at A is $-\frac{m}{AP}$,
nce the magnetic potential at P due
the magnet is

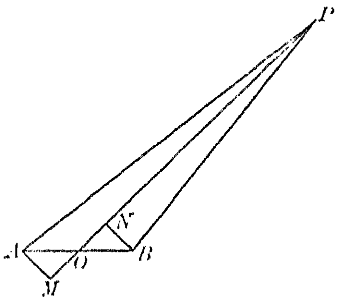


Fig. 57

$$\frac{m}{BP}-\frac{m}{AP}.$$

From A and B let fall perpendiculars AM and BN on OP: since
angles BPO, APO are very small and the angles at M and N are
ht angles, the angles PBN and PAM will be very nearly right
gles, so that approximately

$$BP = PN = PO - ON,$$

$$AP = PM = PO + OM = PO + ON.$$

Then
$$\frac{m}{BP}-\frac{m}{AP}=\frac{m}{PO-ON}-\frac{m}{PO+ON}$$
$$=\frac{2m\,.\,ON}{OP^2-ON^2},$$

l this, since ON is very small compared with OP, is approximately
ual to

$$\frac{2m\,.\,ON}{OP^2}$$
$$=\frac{mAB\cos\theta}{OP^2},$$

ere θ is the angle POB.

If M is the magnetic moment of the magnet

$$M = mAB,$$

hence the potential due to the magnet is equal to

$$\frac{M \cos \theta}{OP^2}.$$

124. Resolution of Small Magnets.

We shall first prove that the moment of a small magnet may be resolved like a force, i.e. if the moment of the magnet is M, and if a force M acting along the axis of the magnet be resolved into forces M_1, M_2, M_3, &c. acting in directions OL_1, OL_2, OL_3, &c., where O is the point midway between the poles, then the magnetic action of the original magnet at a distant point is the same as the combined effects of the magnets whose moments are M_1, M_2, M_3, &c., and whose axes are along OL_1, OL_2, OL_3, &c.

Now suppose a force M in the direction AB, Fig. 57, is the resultant of the forces M_1, M_2, M_3 in the directions OB_1, OB_2, OB_3, &c., let OB_1, OB_2, OB_3 make angles θ_1, θ_2, θ_3 with OP, then

$$M \cos \theta = M_1 \cos \theta_1 + M_2 \cos \theta_2 + \dots,$$

and

$$\frac{M \cos \theta}{OP^2} = \frac{M_1 \cos \theta_1}{OP^2} + \frac{M_2 \cos \theta_2}{OP^2} + \dots.$$

Now $M_1 \cos \theta_1 / OP^2$ is the magnetic potential at P due to the magnet whose moment is M_1 and whose axis is along OB_1, $M_2 \cos \theta_2 / OP^2$ is the potential due to the magnet whose moment is M_2 and whose axis is OB_2, and so on; hence we see that the original magnet may be replaced by a series of magnets, the original moment being the resultant of the moments of the magnets by which the magnet is replaced. In other words, the moment of a small magnet may be resolved like a force.

By the aid of this theorem the problem of finding the force due to a small magnet at any point may be reduced to that of finding the force due to a magnet at a point on its axis produced, and at a point on a line through its centre at right angles to its axis.

125. To find the magnetic force at a point on the axis produced. Let AB, Fig. 58, be the magnet, P the point at

ich the force is required. The magnetic force at P due to the
arge m at B is equal to

$$\frac{m}{(OP - OB)^2}.$$

The magnetic force due to $-m$ at A is equal to

$$-\frac{m}{(OP + OB)^2}.$$

The resultant magnetic force at P is equal to

$$\frac{m}{(OP - OB)^2} - \frac{m}{(OP + OB)^2} = \frac{4m \,.\, OB \,.\, OP}{(OP^2 - OB^2)^2}$$
$$= \frac{4m \,.\, OB \,.\, OP}{OP^4}$$

proximately, since OB is small compared with OP.

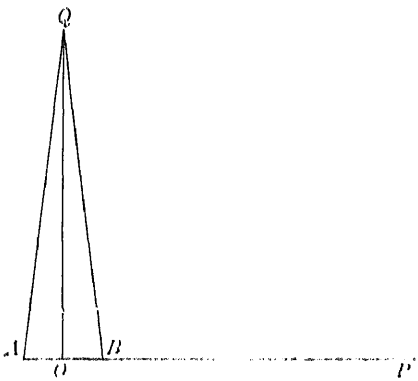


Fig. 58

If M is the moment of the magnet $M = 2mOB$, thus the magnetic
ce at P is equal to

$$\frac{2M}{OP^3}.$$

The direction of this force is along OP.

**126. To find the magnetic force at a point Q on the
e through O at right angles to AB.** Since Q is equidistant
m A and B, Fig. 58, the forces due to A and B are equal in

magnitude; the one being a repulsion, the other an attraction. The resultant of these forces is equal to

$$\frac{2m}{BQ^2}\frac{OB}{BQ}=\frac{M}{BQ^3}$$
$$=\frac{M}{OQ^3},$$

since BQ is approximately equal to OQ.

The direction of this force is parallel to BA and at right angles to OQ.

If Q, a point on the line through O at right angles to AB, is the same distance from O as P, a point on AB produced, we see from these results that the force at P is twice that at Q. This is the foundation of Gauss's method (see Art. 132) of proving that the force between two poles varies inversely as the square of the distance between them.

127. Magnetic force due to a small magnet at any point. Let AB, Fig. 59, represent the small magnet, let M be its moment, O its centre, P the point at which the force is required, let OP make an angle θ with AB, the axis of the magnet. By Art. 124 the effect of M is equivalent to that of two magnets, one having its axis along OP and its moment equal to $M\cos\theta$, the other having its axis at right angles to OP and its moment equal to $M\sin\theta$. Let $OP = r$.

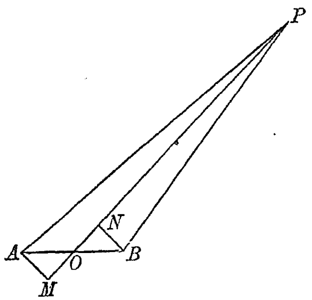


Fig. 59

The force at P due to the first is, by Art. 125, along OP and equal to $2M\cos\theta/r^3$, the force at P due to the second magnet is at right angles to OP and equal to $M\sin\theta/r^3$, hence the force due to the magnet AB at P is equivalent to the forces

$$\frac{2M\cos\theta}{r^3}\text{ along }OP,$$

and $$\frac{M\sin\theta}{r^3}\text{ at right angles to }OP.$$

Let the resultant magnetic force at P make an angle ϕ with OP, n

$$\tan\phi = \frac{\dfrac{M\sin\theta}{r^3}}{\dfrac{2M\cos\theta}{r^3}} = \tfrac{1}{2}\tan\theta.$$

Let the direction of the resultant force at P cut AB produced in draw TL at right angles to OP, then

$$\tan\phi = \frac{TL}{PL}.$$

$$\tan\theta = \frac{TL}{OL},$$

l since $\tan\phi = \frac{1}{2}\tan\theta$, $PL = 2OL$. Thus $OL = \frac{1}{3}OP$. Thus, to l the direction of the magnetic force at P, trisect OP at L, draw ' at right angles to OP to cut AB produced in T, then PT will the direction of the force at P.

The magnitude of the resulting force is

$$\frac{M}{r^3}\sqrt{4\cos^2\theta + \sin^2\theta} = \frac{M}{r^3}\sqrt{1 + 3\cos^2\theta};$$

a given value of r it is greatest when $\theta = 0$ or π, i.e. at a point ng the axis, and least when $\theta = \pi/2$ or $3\pi/2$, i.e. at a point on e line at right angles to the axis. The maximum value is twice e minimum one.

The curves of constant magnetic potential are represented by uations of the form

$$\frac{\cos\theta}{r^2} = C;$$

e lines of force which cut the equipotential curves at right angles e given by the equations

$$r = C\sin^2\theta,$$

iere C is a variable parameter.

The radius of curvature of the line of force at a point P can sily be proved to equal

$$\frac{2r}{3\sin\phi\,(1 + \sin^2\phi)},$$

iere ϕ is the angle the line of force makes with OP. Thus the dius of curvature at points on the line bisecting the magnet at ght angles is one-third of the distance of the point from the magnet.

128. Couple on a Magnet in a Uniform Magnetic Field. If a magnet is placed in a uniform field the couple acting on the magnet, and tending to twist it about a line at right angles both to the axis of the magnet and the force in the external field, is

$$MH \sin \theta,$$

where M is the moment of the magnet, H the force in the uniform field, and θ the angle between the axis of the magnet and the direction of the force.

Let AB be the magnet, the negative pole being at A, the positive one at B. Then if m is the strength of the pole at B, the forces on the magnet are a force mH at B in the direction of the external field and an equal and opposite force at A. These two forces are

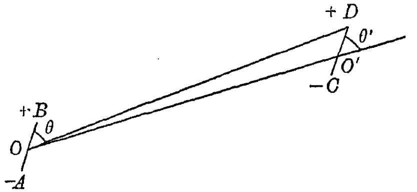


Fig. 60

equivalent to a couple whose moment is $HmNM$, where NM is the distance between the lines of action of the two forces. But

$$NM = AB \sin \theta,$$

if θ is the angle between AB and H; hence the couple on the magnet is

$$HmAB \sin \theta = HM \sin \theta.$$

129. Couples between two Small Magnets.

Let AB, CD, Fig. 60, represent the two magnets; M, M' their moments; r the distance between their centres O, O'. Let AB, CD make respectively the angles θ, θ' with OO'.

Consider first the couple on the magnet CD.

The magnetic forces due to AB are

$$\frac{2M \cos \theta}{r^3} \text{ along } OO',$$

$$\frac{M \sin \theta}{r^3} \text{ at right angles to } OO'.$$

These may be regarded as constant over the space occupied by the small magnet CD.

The couple on CD tending to produce rotation in the direction he hands of a watch, due to the first component, is

$$\frac{2M\cos\theta}{r^3} M' \sin\theta',$$

t due to the second is

$$-\frac{M\sin\theta}{r^3} M' \cos\theta';$$

ice the total couple on CD is

$$\frac{MM'}{r^3}(2\cos\theta\sin\theta' + \sin\theta\cos\theta').$$

This vanishes if $\tan\theta' = -\frac{1}{2}\tan\theta$, i.e. if CD is along the line force due to AB, see Art. 127.

We may show in a similar way that the couple on AB due to tending to produce rotation in the direction of the hands of a tch is

$$\frac{MM'}{r^3}(2\cos\theta'\sin\theta + \sin\theta'\cos\theta).$$

For both these couples to vanish, $\theta = 0$ or π, $\theta' = 0$ or π, or $\pm\frac{\pi}{2}$, $\theta' = \pm\frac{\pi}{2}$, so that the axes of the magnets must be parallel each other, and either parallel or perpendicular to the line joining centres of the two magnets.

We shall find it convenient to consider four special positions of two magnets as standard cases.

CASE I.

Fig. 61

$\theta = 0$, $\theta' = 0$, couples vanish, equilibrium stable.

CASE II.

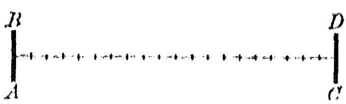


Fig. 62

$\theta = \frac{\pi}{2}$, $\theta' = \frac{\pi}{2}$, couples vanish, equilibrium unstable.

CASE III.

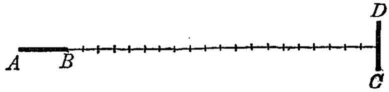


Fig. 63

$$\theta = 0,\ \theta' = \frac{\pi}{2},\ \text{couple on } CD = \frac{2MM'}{r^3},\ \text{couple on } AB = \frac{MM'}{r^3}.$$

When the magnets are arranged as in this case, AB is said to be 'end on' to CD, while CD is 'broadside on' to AB.

CASE IV.

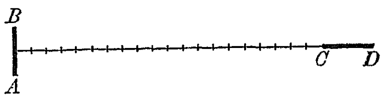


Fig. 64

$$\theta = \frac{\pi}{2},\ \theta' = 0,\ \text{couple on } CD = \frac{MM'}{r^3},\ \text{couple on } AB = \frac{2MM'}{r^3}.$$

In this case AB is broadside on to CD. We see that the couple exerted on CD by AB is twice as great when the latter is end on as when it is broadside on.

It will be noticed that the couples on AB and CD are not in general equal and opposite; at first sight it might appear that this result would lead to the absurd conclusion that if two magnets were firmly fastened to a board, and the board floated on a vessel of water, the board would be set in rotation and would spin round with gradually increasing velocity. The paradox will however be explained if we consider the *forces* exerted by one magnet on the other.

130. Forces between two Small Magnets. Let AB, CD (Fig. 60) represent the two magnets, O, O' the middle points of AB, CD respectively, θ, θ' the angles which AB, CD respectively make with OO'. Let ϕ be the angle DOO', $r = OO'$; m, m' the strengths of the poles of AB and CD.

The force due to the magnet AB on the pole at D consists of the component

$$\frac{2Mm'}{OD^3}\cos(\theta - \phi),$$

long OD, and

$$\frac{Mm'}{OD^3}\sin(\theta-\phi),$$

t right angles to OD.

These are equivalent to a force equal to

$$\frac{2Mm'\cos(\theta-\phi)\cos\phi}{OD^3}+\frac{Mm'\sin(\theta-\phi)\sin\phi}{OD^3},$$

long OO', and a force equal to

$$\frac{2Mm'\cos(\theta-\phi)\sin\phi}{OD^3}-\frac{Mm'\sin(\theta-\phi)\cos\phi}{OD^3},$$

cting upwards at right angles to OO'.

Neglecting squares and higher powers of CD/OO' we have

$$\cos\phi=1,\quad \sin\phi=\frac{CD}{2r}\sin\theta',$$

$$OD=r+\frac{1}{2}CD\cos\theta',\quad \frac{1}{OD^3}=\frac{1}{r^3}-\frac{3}{2}\frac{CD}{r^4}\cos\theta'.$$

Substituting these values we see that the force exerted by AB n D is approximately equivalent to a component

$$\frac{2Mm'\cos\theta}{r^3}-\frac{3Mm'CD\cos\theta\cos\theta'}{r^4}+\frac{3}{2}\frac{Mm'CD\sin\theta\sin\theta'}{r^4},$$

long OO', and a component

$$-\frac{Mm'\sin\theta}{r^3}+\frac{3}{2}\frac{Mm'CD\sin\theta\cos\theta'}{r^4}+\frac{3}{2}\frac{Mm'CD\cos\theta\sin\theta'}{r^4},$$

cting upwards at right angles to OO'.

We may show in a similar way that the force exerted by AB on C is equivalent to a component

$$-\frac{2Mm'\cos\theta}{r^3}-\frac{3Mm'CD\cos\theta\cos\theta'}{r^4}+\frac{3}{2}\frac{Mm'CD\sin\theta\sin\theta'}{r^4},$$

long OO', and a component

$$\frac{Mm'\sin\theta}{r^3}+\frac{3}{2}\frac{Mm'CD\sin\theta\cos\theta'}{r^4}+\frac{3}{2}\frac{Mm'CD\cos\theta\sin\theta'}{r^4}.$$

cting upwards at right angles to OO'.

Hence the force on the magnet CD, which is the resultant of the forces acting on the poles C, D, is equivalent to a component

$$-\frac{3MM'}{r^4}(2\cos\theta\cos\theta' - \sin\theta\sin\theta'),$$

along OO', and a component

$$\frac{3MM'}{r^4}(\sin\theta\cos\theta' + \cos\theta\sin\theta'),$$

acting upwards at right angles to OO'.

The force on the magnet AB is equal in magnitude and opposite in direction to that on CD.

If we consider the two magnets as forming one system, the two forces at right angles to OO' are equivalent to a couple whose moment is

$$\frac{3MM'}{r^3}(\cos\theta\sin\theta' + \sin\theta\cos\theta'),$$

this couple is equal in magnitude and opposite in direction to the algebraical sum of the couples on the magnets AB, CD found in Art. 129: this result explains the paradox alluded to at the end of that article.

131. Force between the Magnets in the four standard positions. In the positions described in Art. 129, the forces between the magnets have the following values.

CASE I. Fig. 61.

$\theta = 0$, $\theta' = 0$. Force between magnets is an attraction along the line joining their centres equal to

$$\frac{6MM'}{r^4}.$$

CASE II. Fig. 62.

$\theta = \frac{\pi}{2}$, $\theta' = \frac{\pi}{2}$. Force is a repulsion along the line joining the centres equal to

$$\frac{3MM'}{r^4}.$$

CASE III. Fig. 63.

$\theta = 0$, $\theta' = \frac{\pi}{2}$. Force is at right angles to the line joining the ntres and equal to

$$\frac{3MM'}{r^4}.$$

CASE IV. Fig. 64.

$\theta = \frac{\pi}{2}$, $\theta' = 0$. Force is at right angles to the line joining the ntres and equal to

$$\frac{3MM'}{r^4}.$$

The forces between the magnets vary inversely as the fourth wer of the distance between their centres, while the couples vary versely as only the cube of this distance. The directive influence hich the magnets exert on each other thus diminishes less quickly th the distance than the translatory forces, so that when the agnets are far apart the directive influence is much the more portant of the two.

132. Gauss's proof that the force between two magnetic les varies inversely as the square of the distance between em. We saw, Art. 129, that, the distance between the magnets naining the same, the couple exerted by the first magnet on the cond was twice as great when the first magnet was 'end on' to e second as when it was 'broadside on.' This is equivalent to the sult proved in Art. 127, that when P and Q are two points at the me distance from the centre of the magnet, P being on the axis the magnet and Q on the line through the centre at right angles the axis, the magnetic force at P is twice that at Q. This result ly holds when the force varies inversely as the square of the stance; we shall proceed to show that if the force varied inversely the pth power of the distance the magnetic force at P would be times that at Q.

If the magnetic force varies inversely as the pth power of the stance, then if m is the strength of one of the poles of the magnet,

the magnetic force at P, Fig. 58, due to the magnet AB is equal to

$$\frac{m}{BP^p} - \frac{m}{AP^p}$$

$$= \frac{m}{(OP - OB)^p} - \frac{m}{(OP + OB)^p}$$

$$= \frac{2mp \, . \, OB}{OP^{p+1}},$$

approximately, if OB is very small compared with OP; if M is the moment of AB this is equal to

$$\frac{pM}{OP^{p+1}}.$$

The force at $$Q = \frac{m}{BQ^p}\frac{OB}{BQ} + \frac{m}{AQ^p}\frac{OA}{AQ}$$

$$= \frac{M}{OP^{p+1}},$$

approximately.

Thus the magnetic force at P is p times that at Q. We see from this that if we have two small magnets the couple on the second when the first magnet is 'end on' to it is p times the couple when the first magnet is 'broadside on.' Hence by comparing the value of the couples in these positions we can determine the value of p.

This can be done by an arrangement of the following kind. Suspend the small magnet which is to be deflected so that it can turn freely about a vertical axis: a convenient way of doing this and one which enables the angular motion of the magnet to be accurately determined, is to place the magnet at the back of a very light mirror and suspend the mirror by a silk fibre. When the deflecting magnet is far away the suspended magnet will under the influence of the earth's magnetic field point magnetic north and south. When this magnet is at rest bring the deflecting magnet into the field and place it so that its centre is due east or west of the centre of the deflected magnet, the axis of the deflecting magnet passing through the centre of this magnet. The couple due to the deflecting magnet will make the suspended magnet swing from the north and south position until the couple with which the earth's magnetic force

to bring the magnet back to its original position just balances
flecting couple.

H be the magnetic force in the horizontal plane due to the
magnetic field. Then when the deflected magnet has twisted
h an angle θ the couple due to the earth's magnetic field is,
t. 128, equal to

$$HM' \sin \theta,$$

M' is the moment of the deflected magnet.

e other magnet may be regarded as producing a field such that
gnetic force at the centre of the deflected magnet is east and
nd equal to

$$\frac{Mp}{r^{p+1}},$$

M is the moment of the deflecting magnet, r the distance
en the centres of the deflected and deflecting magnets. Thus
uple on the deflected magnet due to this magnet is

$$\frac{MM' p \cos \theta}{r^{p+1}}.$$

spended magnet will take up the position in which the two
s balance: when this is the case

$$HM' \sin \theta = \frac{MM' p \cos \theta}{r^{p+1}},$$

$$\tan \theta = \frac{Mp}{Hr^{p+1}} \text{ ....(1).}$$

w place the deflecting magnet so that its centre is north or
of that of the suspended magnet, and at the same distance
t as in the last experiment, the axis of the deflecting magnet
again east and west. Let the suspended magnet be in equili-
when it has twisted through an angle θ'. The couple due to
rth's magnetic field is

$$HM' \sin \theta'.$$

e couple due to the deflecting magnet is

$$\frac{MM' \cos \theta'}{r^{p+1}}.$$

Since the suspended magnet is in equilibrium these couples must be equal, hence

$$HM' \sin\theta' = \frac{MM' \cos\theta'}{r^{p+1}},$$

hence
$$\tan\theta' = \frac{M}{Hr^{p+1}} \quad\text{.........................(2).}$$

Thus
$$\frac{\tan\theta}{\tan\theta'} = p.$$

Hence if we measure θ and θ' we can determine p. By experiments of this kind Gauss showed that $p = 2$, i.e. that the force between two poles varies inversely as the square of the distance between them.

If we place the deflecting magnet at different distances from the deflected we find that $\tan\theta$ and $\tan\theta'$ vary as $1/r^3$, and thus obtain another proof that $p = 2$.

133. Determination of the Moment of a Small Magnet and of the horizontal component of the Earth's Magnetic Force. Suspend a small auxiliary magnet in the same way as the deflected magnet in the experiment just described, and place the magnet A whose moment is to be determined, so that its centre is due east or west of the centre of the auxiliary magnet, and its axis passes through the centre of the suspended magnet. Let θ be the deflection of the suspended magnet, H the horizontal component of the earth's magnetic force, M the moment of A: we have, by equation (1), Art. 132, putting $p = 2$

$$\frac{M}{H} = \tfrac{1}{2} r^3 \tan\theta;$$

hence if we measure r and θ we can determine M/H.

To determine MH suspend the magnet A so that it can rotate freely about a vertical axis, passing through its centre, taking care that the magnetic axis of A is horizontal. When the magnet makes an angle θ with the direction in which H acts, i.e. with the north and south line, the couple tending to bring it back to its position of equilibrium is equal to

$$MH \sin\theta.$$

ence if K is the moment of inertia of the magnet about the al axis the equation of motion of the magnet is

$$K\frac{d^2\theta}{dt^2}+MH\sin\theta=0,$$

is small

$$K\frac{d^2\theta}{dt^2}+MH\theta=0.$$

T, the time of a small oscillation, is given by the equation

$$T=2\pi\sqrt{\frac{K}{MH}},$$

$$MH=\frac{4\pi^2K}{T^2},$$

if we know K and T we can determine MH; and knowing from the preceding experiment we can find both M and H. alue of H at Cambridge is about ·18 C.G.S. units.

4. **Magnetic Shell of Uniform Strength.** A magnetic is a thin sheet of magnetizable substance magnetized at point in the direction of the normal to the sheet at that

e **strength** of the shell at any point is the product of the ity of magnetization into the thickness of the shell measured the normal at that point, it is thus equal to the magnetic nt of unit area of the shell at the point.

find the potential of a shell of uniform strength. Consider ll area α of the shell round the Q, Fig. 65, let I be the intensity gnetization of the shell at Q, hickness of the shell at the same The moment of the small magnet area is α is $I\alpha t$, hence if θ is the which the direction of magnetiza- makes with PQ, the potential of nall magnet at P is by Art. 123 equal to

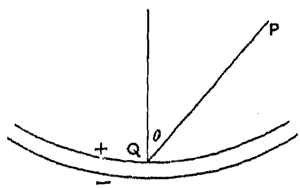


Fig. 65

$$\frac{I\alpha t\cos\theta}{PQ^2}.$$

If ϕ is the strength of the magnetic shell

$$It = \phi,$$

hence the potential at P is

$$\frac{\phi\alpha \cos\theta}{PQ^2}.$$

This, by Art. 10, is numerically equal to the normal induction over α due to a charge of electricity equal to ϕ at P. Hence if ϕ is constant over the shell the potential of the whole shell at P is numerically equal to the total normal electric induction over it due to a charge ϕ at P. This, by Art. 10, is equal to $\phi\omega$, where ω is the area cut off from the surface of a sphere of unit radius with its centre at P by lines drawn from P to the boundary of the shell; ω is called the solid angle subtended by the shell at P; it only depends on the shape of the boundary of the shell.

If the shell is closed, then if P is outside the shell the potential at P is zero, since the total normal electric induction over a closed surface due to a charge at a point outside the surface is zero; if the point P is inside the surface and the negative side of the shell is on the outside, then since the total normal electric induction over the shell due to a charge ϕ at P is $4\pi\phi$, the magnetic potential at P is $4\pi\phi$; as this is constant throughout the shell, the magnetic force vanishes inside the space bounded by the shell.

The signs to be ascribed to the solid angle bounded by the shell at various points are determined in the following way. Take a fixed point O and with it as centre describe a sphere of unit radius. Let P be a point at which the magnetic potential of the shell is required. The contribution to the magnetic potential by any small area round a point Q on the shell, is the area cut off from the surface of the sphere of unit radius by the radii drawn from O parallel to the radii drawn from P to the boundary of the area round Q. The area enclosed by the lines from O is to be taken as positive or negative according as the lines drawn from P to Q strike first against the positive or negative side of the shell. By the positive side of the shell we mean the side charged with positive magnetism, by the negative side the side charged with negative magnetism.

With this convention with regard to the signs of the solid angle, let us consider the relation between the potentials due to a shell at

two points P and P'; P being close to the shell on the positive side, P' close to P but on the negative side of the shell. Consider the areas traced out on the unit sphere by radii from O parallel to those drawn from P and P'. The area corresponding to those drawn from P will be the shaded part of the sphere, let this area be ω, the potential at P is $\phi\omega$. The area corresponding to the radii drawn from P' will be the unshaded portion of the sphere whose area is $4\pi - \omega$, but inasmuch as the radii from P' strike first against the negative side of the shell the solid angle subtended at P' will be minus this area, i.e. $\omega - 4\pi$; hence the magnetic potential due to the shell at P' is $\phi(\omega - 4\pi)$. The potential at P thus exceeds that at P' by $4\pi\phi$.

Fig. 66

In spite of this finite increment in the potential in passing from P' to the adjacent point P, there will be continuity of potential in passing through the shell if we regard the potential as given in the shell by the same laws as outside.

Consider the potential at a point Q in the shell, and divide the original shell into two, one on each side of Q. Then as the whole shell is uniformly magnetized the strength of the shells will be proportional to their thicknesses. Thus if ϕ is the strength of the original shell the strength of the shell between P and Q will be $\phi\,\dfrac{PQ}{PP'}$, and that of the shell between Q and P' will be $\dfrac{QP'}{PP'}$.

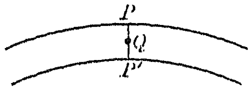


Fig. 67

The potential at Q due to the shell next to P' is $\omega\phi\,\dfrac{QP'}{PP'}$, that due to the shell next to P is $(\omega - 4\pi)\,\phi\,\dfrac{QP}{PP'}$, the potential at Q is the sum of these, i.e.

$$\omega\phi - 4\pi\phi\,\frac{QP}{PP'};$$

this changes continuously as we pass through the shell from

$$\phi\,(\omega - 4\pi) \text{ at } P',$$

to

$$\phi\omega \text{ at } P.$$

135. Mutual Potential Energy of the Shell and an external Magnetic System. Let I be the intensity of magnetization at a point Q on the shell; consider a small portion of the shell round Q, α being the area of this portion. Let P, P' be two points on its axis of magnetization, P being on the positive surface of the shell, P' on the negative. Then we have a charge of positive magnetism equal to $I\alpha$ at P, a negative charge $-I\alpha$ at P'. If V_P, $V_{P'}$ are the potentials at P and P' respectively due to the external magnetic system, then the mutual potential energy of the external system and the small magnet at Q is equal to

$$V_P I\alpha - V_{P'} I\alpha \quad \text{.......................(1).}$$

If ϕ is the strength of the shell

$$\phi = I \times PP',$$

hence the expression (1) is equal to

$$\frac{\phi\,(V_P - V_{P'})\,\alpha}{PP'}.$$

But $(V_P - V_{P'})/PP'$ is the magnetic force due to the external system along PP', the normal to the shell. Let this force be denoted by $-H_n$, the force being taken as positive when it is in the direction of magnetization of the shell, i.e. when the magnetic force passes from the negative to the positive side through the shell, then the mutual potential energy of the external system and the small magnet at Q is equal to

$$-\phi H_n \alpha.$$

Since the strength of the shell is uniform the mutual potential energy of the external system and the whole shell is equal to

$$-\phi\Sigma H_n \alpha,$$

$\Sigma H_n \alpha$ being the sum of the products got by dividing the surface of the shell up into small areas, and multiplying each area by the component along its normal of the magnetic force due to the external system, this component being positive when it is in the direction of magnetization of the shell. This quantity is often called

mber of lines of magnetic force due to the external system pass through the shell.

s analogous to the total normal electric induction over a surface ctrostatics, see Art. 9.

6. Force acting on the shell when placed in a mag- field. If X is the force acting on the shell in the direction l if the shell is displaced in this on through a distance δx, then $X\delta x$ work done on the shell by the tic forces during the displacement; by the principle of the Conservation rgy, $X\delta x$ must equal the diminution energy due to the displacement. se that A, Fig. 68, represents the n of the edge of the shell before, position after the displacement. iminution in the energy due to the cement is, by the last paragraph, to

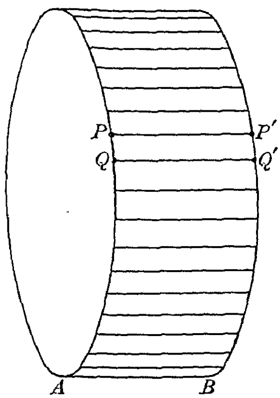


Fig. 68

$$\phi\,(N' - N)\;\ldots\ldots\ldots\ldots(1),$$

N and N' are the numbers of lines of magnetic force which through A and B respectively. Consider the closed surface g as ends the shell in its two positions A and B, the sides of rface being formed by the lines PP' &c. which join the original on of a point P to its displaced position. We see, as in Art. 10, unless the closed surface contains an excess of magnetism of gn $\Sigma H_n a$ taken over its surface must vanish, H_n denoting the etic force along the normal to the surface drawn outwards.

ut $\Sigma H_n a$ over the whole surface

$$= N' - N + \Sigma H_n a \text{ taken over the sides,}$$

$$N' - N = -\Sigma H_n a \;\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

ummation on the right-hand side of this equation being taken the sides. Consider a portion of the sides bounded by PQ, $P'Q'$; ′ being the displaced positions of P and Q respectively. Since

$$PP' = QQ' = \delta x,$$

rea $PQP'Q'$ is equal to

$$\delta x \times PQ \times \sin\theta,$$

where θ is the angle between PQ and PP'. If H is the magnetic force at P due to the external system, the value of $H_n\alpha$ for the element $PQQ'P'$ is equal to

$$\delta x \times PQ \times \sin\theta \times H \cos\chi,$$

where χ is the angle which the outward-drawn normal to $PQQ'P'$ makes with H. Hence since $X\delta x = \phi\,(N' - N)$ we have by equation (2)

$$X\delta x = -\phi\Sigma\{\delta x \times PQ \times \sin\theta \times H\cos\chi\},$$

or since δx is the same for all points on the shell

$$X = -\phi\Sigma\{PQ \times \sin\theta \times H\cos\chi\}.$$

Thus the force on the shell parallel to x is the same as it would be if a force parallel to x acted on the boundary of the shell, equal per unit length to

$$-\phi H\sin\theta\cos\chi.$$

Since x is arbitrary this gives the force acting on each element of the boundary in any direction; to find the resultant force on the element, we notice that the component along x vanishes if x is parallel to PQ, for in this case $\theta = 0$, the resultant force is thus at right angles to the element of the boundary. Again, if x is parallel to H, $\chi = \pi/2$, and the force again vanishes, thus the resultant force is at right angles to H. Hence the resultant force on PQ is at right angles both to PQ and H. In order to find the magnitude of this force we have only to suppose that x is parallel to this normal, in this case $\theta = \pi/2$ and $\chi = \dfrac{\pi}{2} - \psi$, where ψ is the angle between PQ and H; the resultant force is therefore

$$-\phi H\sin\psi.$$

Thus the force on the shell may be regarded as equivalent to a system of forces acting over the edge of the shell, the force acting on each element of the edge being at right angles to the element and to the external magnetic force at the element, and equal per unit length to the product of the strength of the shell into the component of the magnetic force at right angles to the element of the edge.

The preceding rule gives the line along which the force acts; the direction of the force is, in any particular case, most easily got from the principle that since the mutual potential energy of the shell and

ternal magnetic system is equal to $-\phi N$, where N is the er of lines of magnetic force due to the external system which hrough the shell in the direction in which it is magnetized, hich enter the shell on the side with the negative magnetic and leave it on the side with the positive charge: the shell end to move so as to make N as large as possible, for by so it makes the potential energy as small as possible. The force h element of the boundary will therefore be in such a direction tend to move the element of the boundary so as to enclose ter number of lines of magnetic force passing through the shell positive direction.

us if the direction of the magnetic force at the element PQ is direction PT in Fig. 69, the force on ll be outwards along PS as in the figure, PQ were to move in this direction the would catch more lines of force passing gh it in the positive direction.

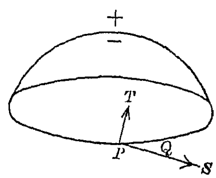


Fig. 69

nce $$X\,\delta x = \phi\,(N' - N)$$

$$X = \phi\,\frac{dN}{dx}.$$

is expression is often very useful for finding the total force on ell in any direction.

37. Magnetic force due to the shell. Suppose that the al field is that due to a single unit pole at a point A, the result preceding article will give the force on the shell due to the this must however be equal and opposite to the force exerted e shell on the pole. If however the field is due to a unit pole H the magnetic force due to the external system at an element the shell is equal to $1/AP^2$ and acts along AP: hence by the rticle the magnetic force at A due to the shell is the same as supposed each unit of length of the boundary of the shell to a force equal to

$$\frac{\phi}{AP^2}\sin\theta,$$

θ is the angle between AP and the tangent to the boundary ϕ is the strength of the shell. This force acts along the line is at right angles both to AP and the tangent to the boundary

at P. The direction in which the force acts along this line may be found by the rule that it is opposite to the force acting on the element of the boundary at P arising from unit magnetic pole at A; this latter force may be found by the method given at the end of the preceding article.

138. If the external magnetic field in Art. 135 is due to a second magnetic shell, then the mutual potential energy of the two shells is equal to

$$-\phi N,$$

where ϕ is the strength of the first shell, and N the number of lines of force which pass through the first shell, and are produced by the second. It is also equal to

$$-\phi' N',$$

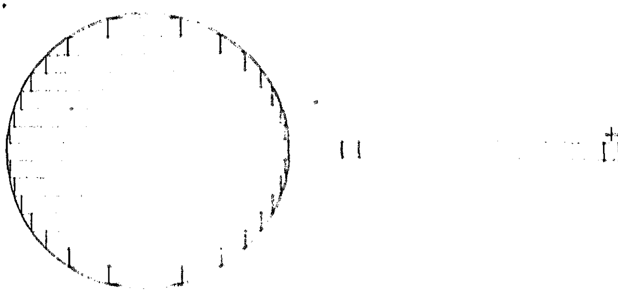


Fig. 70

where ϕ' is the strength of the second shell, and N' the number of lines of force which pass through the second shell, and are produced by the first. Hence by making $\phi = \phi'$ we see that, if we have two shells α and β of equal strengths, the number of lines of force which pass through α and are due to β is equal to the number of lines of force which pass through β and are due to α.

139. Magnetic Field due to a uniformly magnetized sphere. Let the sphere be magnetized parallel to x, and let I be the intensity of magnetization. We may regard the sphere as made up, as in Fig. 70, of a great number of uniformly magnetized bar magnets of uniform cross section a, the axes of these magnets being parallel to the axis of x. On the ends of each of these magnets we have charges of magnetism equal to $\pm Ia$. Now consider a sphere whose radius is equal to that of the magnetized sphere and built up of bars in the same way, each of these bars being however wholly

h positive magnetism whose volume density is ρ: consider ther equal sphere divided up into bars in the same way, these bars being however filled with negative magnetism olume density is $-\rho$; suppose that these spheres have their t O' and O, Fig. 71, two points very close together, OO' being to the axis of x. Consider now lt of superposing these two take two corresponding bars; s of the bars which coincide tralize each other's effects, but tive bar will project a distance the left, and on this part of there will be a charge of negagnetism equal to $OO' \times \alpha \times \rho$: tive bar will project a distance

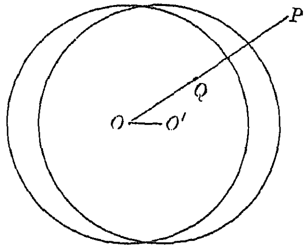


Fig. 71

he right, and on this part of the bar there will be a charge ive magnetism equal to $OO' \times \alpha \times \rho$. If OO' is very small regard these charges as concentrated at the ends of the that if $OO' \times \rho = I$ the case will coincide with that of the ly magnetized sphere.

an easily find the effects of the positive and negative spheres oint either inside or outside. Let us first consider the effect ternal point P.

potential due to the positive sphere is equal to

$$\frac{4}{3}\frac{\pi a^3 \rho}{O'P}$$

e radius of the sphere.

potential due to the negative sphere is equal to

$$-\frac{4}{3}\frac{\pi a^3 \rho}{OP}.$$

e the potential due to the combination of the spheres is

$$\frac{4}{3}\pi a^3 \rho \left\{\frac{1}{O'P} - \frac{1}{OP}\right\}$$

$$= \frac{4}{3}\pi a^3 \rho \cdot \frac{OO' \cos\theta}{OP^2}$$

nately, if OO' is very small, and θ is the angle which OP ith OO'.

Now we have seen that this case coincides with that of the uniformly magnetized sphere if $\rho \times OO' = I$, where I is the intensity of magnetization of the sphere; hence the potential due to the uniformly magnetized sphere at an external point P is

$$\frac{4}{3}\pi a^3 I \cdot \frac{\cos\theta}{r^2},$$

where $r = OP$.

Comparing this result with that given in Art. 123 we see that the uniformly magnetized sphere produces the same effect outside the sphere as a very small magnet placed at its centre, the axis of the small magnet being parallel to the direction of magnetization of the sphere, while the moment of the magnet is equal to the intensity of magnetization multiplied by the volume of the sphere.

The magnetic force inside the sphere is indefinite without further definition, since to measure the force on the unit pole, we have to make a hole to receive the pole and the force on the pole depends on the shape of the hole so made: this point is discussed at length in Chapter VIII.

For the sake of completing the solution of this case, we shall anticipate the results of that chapter and assume that the quantity which is defined as 'the magnetic force' inside the sphere is the force which would be exerted on the unit pole if the sphere were regarded as a spherical air cavity over the surface of which there is spread the same distribution of magnetic charge as actually exists over the surface of the magnetized sphere. We may thus in calculating the effect of the charges on the surface suppose that they exert the same magnetic forces as they would in air.

To find the magnetic force at an internal point Q, Fig. 71, we return to the case of the two uniformly charged spheres.

The force due to the uniformly positively charged sphere at Q is equal to

$$\tfrac{4}{3}\pi\rho \cdot O'Q,$$

and acts along $O'Q$; the force due to the negatively charged sphere is equal to

$$\tfrac{4}{3}\pi\rho \cdot OQ,$$

and acts along QO.

By the triangle of forces the resultant of the forces exerted by the positive and negative spheres is equal to

$$\tfrac{4}{3}\pi\rho \cdot OO',$$

d is parallel to OO'. We have seen that the case of the positive d negative spheres coincides with that of the uniformly magnetized here if $OO' \times \rho = I$. Hence the force inside the uniformly magized sphere is uniform and parallel to the direction of magnetizan of the sphere and equal to

$$\tfrac{4}{3}\pi I.$$

The lines of force inside and outside the sphere are given in g. 72.

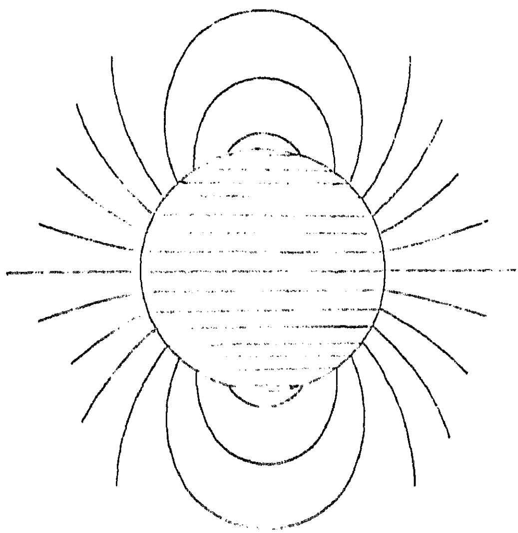


Fig. 72

CHAPTER VII

TERRESTRIAL MAGNETISM

140. The pointing of the compass in a definite direction was at first ascribed to the special attraction for iron possessed by the pole star. Gilbert, however, in his work *De Magnete*, published in 1600, pointed out that it showed that the earth was itself a magnet. Since Gilbert's time the study of Terrestrial Magnetism, i.e. the state of the earth's magnetic field, has received a great deal of attention and forms one of the most important, and undoubtedly one of the most mysterious departments of Physical Science.

141. To fix the state of the earth's magnetic field we require to know the magnetic force over the whole of the surface of the earth; the observations made at a number of magnetic observatories, scattered unfortunately somewhat irregularly at very wide intervals over the earth, give us an approximation to this.

To determine the magnitude and direction of the earth's magnetic force we require to know three things: the three usually taken are (1) the magnitude of the horizontal component of the earth's magnetic force, usually called the earth's horizontal force; (2) the angle which the direction of the horizontal force makes with the geographical meridian, this angle is called the declination; the vertical plane through the direction of the earth's horizontal force is called the magnetic meridian; (3) the dip, that is the complement of the angle which the axis of a magnet, suspended so as to be able to turn freely about an axle through its centre of gravity at right angles to the magnetic meridian, makes with the vertical. The fact that a compass needle when free to turn about a horizontal axis would not settle in a horizontal position, but 'dipped,' so that the north end pointed downwards, was discovered by Norman in 1576.

For a full description of the methods and precautions which must be taken to determine accurately the values of the magnetic elements the student is referred to the article on Terrestrial Magnetism in the *Encyclopædia Britannica*: we shall in what follows merely give a

ral account of these methods without entering into the details
h must be attended to if the most accurate results are to be
ined.

The method of determining the horizontal force has been described
rt. 133.

142. **Declination.** To determine the declination an instrument called a declinometer may be employed; this instrument presented in Fig. 73. The magnet—which is a hollow tube with

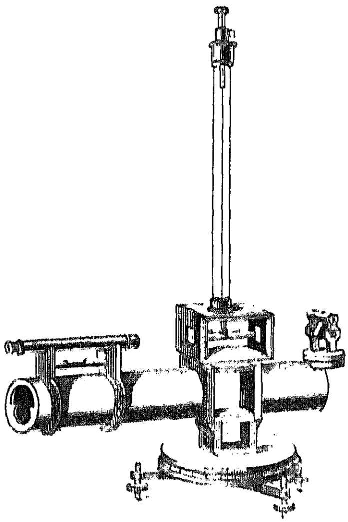


Fig. 73

ece of plane glass with a scale engraved on it at the north end a lens at the south end is suspended by a single long silk thread which the torsion has been removed by suspending from it ummet of the same weight as the magnet: the suspension and reading telescope can rotate about a vertical axis and the azimuth e system determined by means of a scale engraved on the fixed zontal base. The observer looks through the telescope and rves the division on the scale at the end of the magnet with h a cross wire in the telescope coincides; the magnet is then ed upside down and resuspended and the division of the scale which the cross wire coincides again noted; this is done to

correct for the error that would otherwise ensue if the magnetic axis of the cylinder did not coincide with the geometrical axis. The mean of the readings gives the position of the magnetic axis. If now we take the reading on the graduated circle and add to this the known value in terms of the graduations on this circle of the scale divisions seen through the telescope, we shall find the circle reading which corresponds to the magnetic meridian. Now remove the magnet and

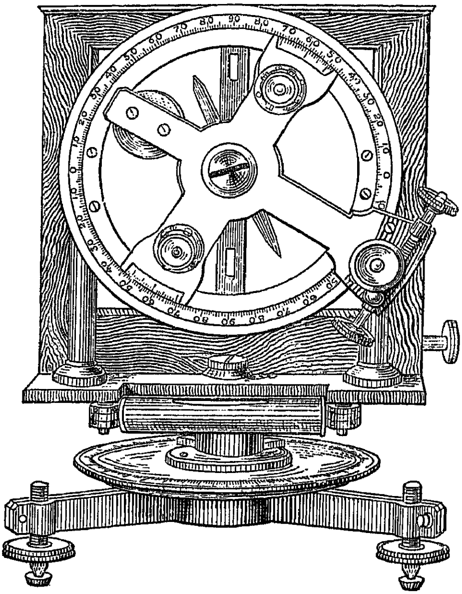


Fig. 74

turn the telescope round until some distant object, whose azimuth is known, is in the field of view; take the reading on the graduated circle, the difference between this and the previous reading will give us the angular distance of the magnetic meridian from a plane whose azimuth is known: in other words, it gives us the magnetic declination.

143. Dip. The dip is determined by means of an instrument called the dip-circle, represented in Fig. 74. It consists of a thin magnet with an axle of hard steel whose axis is at right angles to the plane of the magnet, and ought to pass through the centre of

avity of the needle; this axle rests in a horizontal position on two gate edges, and the angle the needle makes with the vertical is read f by means of the vertical circle. The needle and the vertical circle n turn about a vertical axis. To set the plane of motion of the

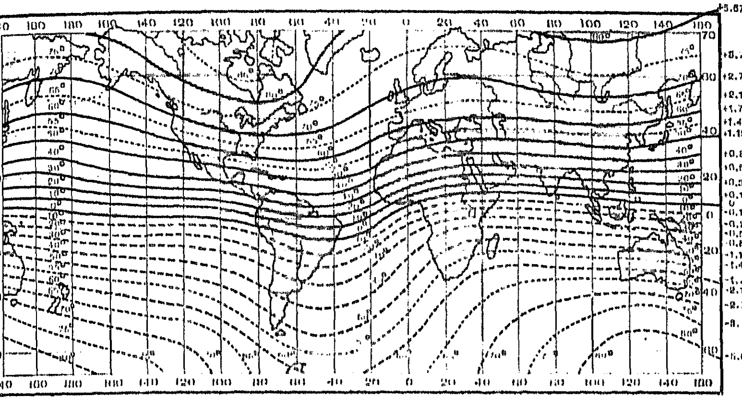


Fig. 75

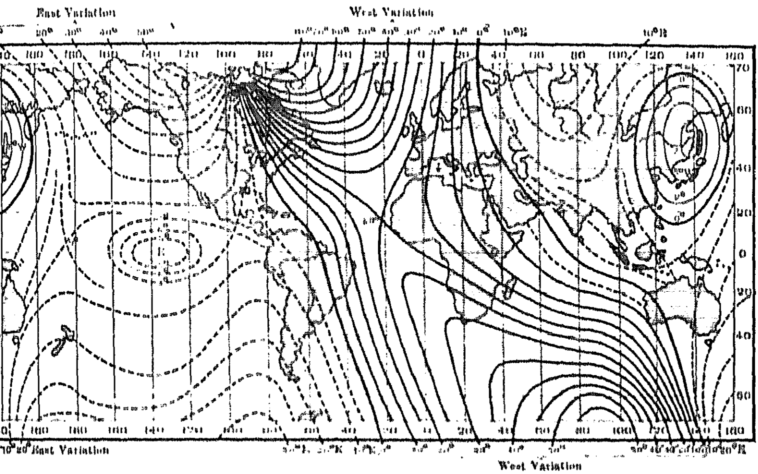


Fig. 76

eedle in the magnetic meridian, the plane of the needle is turned bout the vertical axis until the magnet stands exactly vertical; hen in this position the plane of the needle must be at right angles the magnetic meridian. The instrument is then twisted through

90° (measured on the horizontal circle) and the magnet is then in the magnetic meridian; the angle it makes with the horizontal in this position is the dip. To avoid the error arising from the axle of the needle not being coincident with the centre of the vertical circle, the positions of the two ends of the needle are read; to avoid the error due to the magnetic axis not being coincident with the line joining the ends of the needle, the needle is reversed so that the face which originally was to the east is now to the west and a fresh set of readings taken; and to avoid the errors which would arise if the

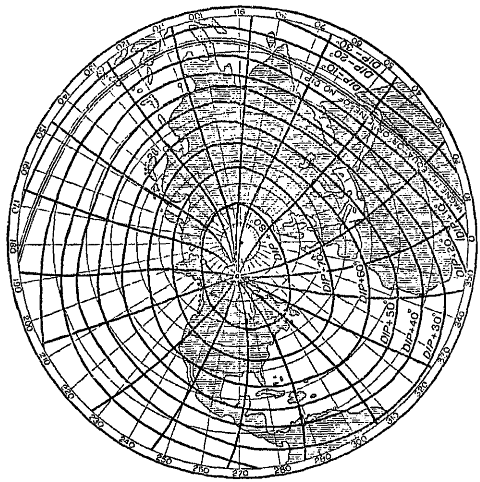


Fig. 77

centre of gravity were not on the axle, the needle is remagnetized so that the end which was previously north is now south and a fresh set of readings taken. The mean of these readings gives the dip.

144. We can embody in the form of charts the determinations of these elements made at the various magnetic observatories: thus, for example, we can draw a series of lines over the map of the world such that all points on one of these lines have the same declination, these are called isogonic lines: we may also draw another set of lines so that all the places on a line have the same dip, these are called isoclinic lines. The lines however which give the best general idea of the distribution of magnetic force over the earth's surface are the lines of horizontal magnetic force on the earth's surface, i.e. the

; which would be traced out by a traveller starting from any t and always travelling in the direction in which the compass ted; they were first used by Duperrey in 1836.

The isoclinic lines, the isogonic lines and Duperrey's lines for the thern and Southern Hemispheres for 1876 are shown in Figs. 75, 77, and 78 respectively.

145. The points to which Duperrey's lines of force converge are ed 'poles,' they are places where the horizontal force vanishes, ; is where the needle if freely suspended would place itself in a ical position.

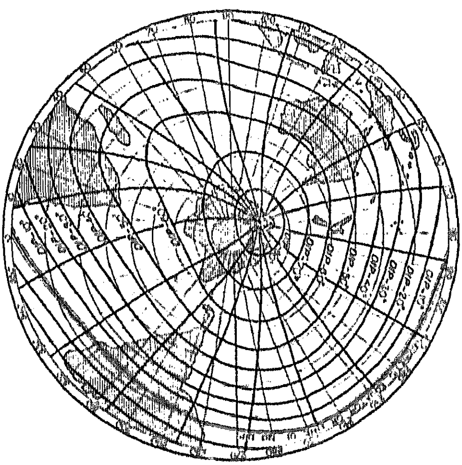


Fig. 78

Gauss by a very thorough and laborious reduction of magnetic ervations gave as the position in 1836, of the pole in the Northern nisphere, latitude 70° 35′, longitude 262° 1′ E., and of the pole he Southern Hemisphere, latitude 78° 35′, longitude 150° 10′ E. poles are thus not nearly at opposite ends of a diameter of the h.

146. An approximation, though only a very rough one, to the e of the earth's magnetic field, may be got by regarding the h as a uniformly magnetized sphere.

On this supposition, we have by Art. 139, if θ is the dip at any ce, i.e. the complement of the angle between the magnetic force

and the line joining the place to the centre of the earth, l the magnetic latitude, i.e. the complement of the angle this line makes with the direction of magnetization of the sphere,

$$\tan\theta = 2\tan l,$$

while the resultant magnetic force would vary as

$$[1 + 3\sin^2 l]^{\frac{1}{2}}.$$

These are only very rough approximations to the truth but are sometimes useful when more accurate knowledge of the magnetic elements is not available.

If M is the moment of the uniformly magnetized sphere which most nearly represents the earth's magnetic field, then in C.G.S. units

$$M = \cdot 323\ (\text{earth's radius})^3.$$

147. Variations in the Magnetic Elements. During the time within which observations of the magnetic elements have been carried on the declination at London has changed from being 11° 15′ to the East of North as in 1580 to 24° 38′ 25″ to the West of North as in 1818. It is now going back again to the East, but is still pointing between 14° and 15° to the West. The variations in the declination and dip in London are shown in the following table:

Date	Declination	Dip
1576		71° 50′
1580	11° 15′ E.	
1600		72° 0′
1622	6° 0′ E.	
1634	4° 6′ E.	
1657	0° 0′ E.	
1665	1° 22′ W.	
1672	2° 30′ W.	
1676		73° 30′
1692	6° 30′ W.	
1723	14° 17′ W.	74° 42′
1748	17° 40′ W.	
1773	21° 9′ W.	72° 19′
1787	23° 19′ W.	72° 8′
1795	23° 57′ W.	
1802	24° 6′ W.	70° 36′

Date	Declination	Dip
1820	24° 34′ 30″ W.	70° 3′
1830	24°	69° 38′
1838		69° 17′
1860	21° 39′ 51″	68° 19′·29
1870	29° 18′ 52″	67° 57′·98
1880	18° 57′ 59″	
1893	17° 27′	67° 30′
1900	16° 52′·7	
1919	14° 18′·2	66° 53′·6

This slow change in the magnetic elements is often called the secular variation. The points of zero declination seem to travel westward.

The rates at which these changes take place show considerable variations. The following table gives the mean annual change at Kew in the Declination D and Dip I.

	D.	I.
1890–1895	− 6′·8	− 1′·8
1895–1900	− 4′·8	− 2′·7
1900–1905	− 4′·0	− 1′·6
1905–1910	− 5′·9	− 1′·0

148. Besides these slow changes in the earth's magnetic force, there are other changes which take place with much greater rapidity.

Diurnal Variation. A freely suspended magnetic needle does not point continually in one direction during the whole of the day. In England in the night from about 7 p.m. to 10 a.m. it points *to the East of magnetic* North and South (i.e. to the East of the mean position of the needle), and during the day from 10 a.m. to 7 p.m. to the West of magnetic North and South. It reaches the westerly limit about 2 in the afternoon, its easterly one about 8 in the morning, the arc travelled over by the compass being about 10 minutes. This arc varies however with the time of the year, being greatest at midsummer and least at midwinter. There are two maxima in summer, one minimum in winter.

The diurnal variation changes very much from one place to another, it is exceedingly small at Trevandrum, a place near the equator, on the other hand in Arctic and Antarctic regions the variations are very large.

the morning.

In the following diagram, due to Prof. Lloyd, the radius vector represents the disturbing force acting on the magnet at different times of the day in Dublin, the forces at any hour are the average of those at that hour for the year. The curve would be different for different seasons of the year.

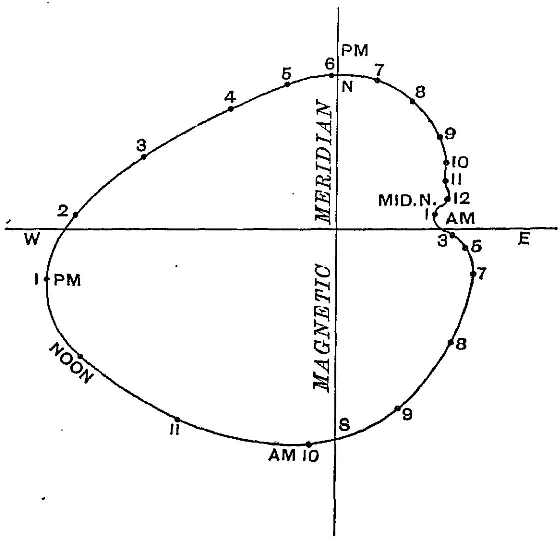


Fig. 79

There is also a diurnal variation in the vertical component of the earth's magnetic force. In England the vertical force is least between 10 and 11 a.m., greatest at about 6 p.m.

The extent of the diurnal variation depends upon the condition of the sun's surface, being greater when there are many sun spots. As the state of the sun with regard to sun spots is periodic, going through a cycle in about eleven years, there is an eleven-yearly period in the magnitude of the diurnal variation.

149. Effect of the Moon. The magnetic declination shows a variation depending on the position of the moon with respect to

eridian, the nature of this variation varies very much in
ent localities.

50. Magnetic Disturbances. In addition to the periodic
egular disturbances previously described, rapid and irregular
es in the earth's magnetic field, called magnetic storms,
ently take place; these often occur simultaneously over a large
on of the earth's surface.

urorae are mostly accompanied by magnetic storms, and there
ry strong evidence that a magnetic storm accompanies the
n formation of a sun spot.

51. Very important evidence as to the locality of the origin of
arth's magnetic field, or of its variations, is afforded by a method
o Gauss which enables us to determine whether the earth's
etic field arises from a magnetic system above or below the
ce of the earth. The complete discussion of this method requires
se of Spherical Harmonic Analyses. The principle underlying
wever can be illustrated by considering a simple case, that of
formly magnetized sphere.

et PQ be two points on a spherical surface concentric with the
e, then by observation of the horizontal force at a series of
ons between P and Q, we can determine the difference between
agnetic potential at P and Q. If Ω_P and Ω_Q are the magnetic
tials at P and Q respectively these observations will give us
Ω_Q. By Art. 139 if θ_1, θ_2 are the angles OP and OQ make with
irection of magnetization of the sphere

$$\Omega_P - \Omega_Q = \frac{M}{r^2}(\cos\theta_1 - \cos\theta_2) \quad \ldots\ldots\ldots\ldots(1),$$

e M is the magnetic moment of the sphere and

$$r = OP = OQ,$$

e O is the centre of the sphere.

Z_P, Z_Q are the vertical components of the earth's magnetic
, i.e. the forces in the direction OP and OQ respectively, then

$$\left.\begin{aligned} Z_P &= \frac{2M}{r^3}\cos\theta_1 \\ Z_Q &= \frac{2M}{r^3}\cos\theta_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

Z_P and Z_Q can of course be determined by observations made at P and Q. By equations (1) and (2), we have

$$\Omega_P - \Omega_Q = \tfrac{1}{2}(Z_P - Z_Q)\, r \quad \text{..................(3)},$$

hence if the field over the surface of the sphere through P and Q were due to an internal uniformly magnetized sphere, the relation (3) would exist between the horizontal and vertical components of the earth's magnetic force.

Now suppose that P and Q are points inside a uniformly magnetized sphere, the force inside the sphere is uniform and parallel to the direction of magnetization, let H be the value of this force, then in this case

$$\begin{aligned} \Omega_P - \Omega_Q &= Hr(\cos\theta_2 - \cos\theta_1), \\ Z_P &= -H\cos\theta_1, \\ Z_Q &= -H\cos\theta_2, \end{aligned}$$

hence in this case

$$\Omega_P - \Omega_Q = -r(Z_P - Z_Q) \quad \text{..................(4)}.$$

Thus if the magnetic system were above the places at which the elements of the magnetic field were determined, the relation (4) would exist between the horizontal and vertical components of the earth's magnetic force. Conversely if we found that relation (3) existed between these components we should conclude that the magnets producing the field were below the surface of the earth, while if relation (4) existed we should conclude the magnets were above the surface of the earth: if neither of these relations was true we should conclude that the magnets were partly above and partly below the surface of the earth.

Gauss showed that no appreciable part of the *mean* values of the magnetic elements was due to causes above the surface of the earth. Schuster has however shown by the application of the same method that the *diurnal variation* must be largely due to such causes. Balfour Stewart had previously suggested the magnetic action of electric currents flowing through rarefied air in the upper regions of the earth's atmosphere as the probable cause of this variation.

CHAPTER VIII

MAGNETIC INDUCTION

152. When a piece of unmagnetized iron is placed in a magnetic field it becomes a magnet, and is able to attract iron filings; it is then said to be magnetized by *induction*. Thus if a piece of soft iron (a common nail for example) is placed against a magnet, it becomes magnetized by induction, and is able to support another nail, while this nail can support another one, and so on until a long string of nails may be supported by the magnet.

If the positive pole of a bar magnet be brought near to one end of a piece of soft iron, that end will become charged with negative magnetism, while the remote end of the piece of iron will be charged with positive magnetism. Thus the opposite poles of these two magnets are nearest each other, and there will therefore be an attraction between them, so that the piece of iron, if free to move, will move towards the inducing magnet, i.e. it will move from the weak to the strong parts of the magnetic field due to this magnet. If, instead of iron, pieces of nickel or cobalt are used they will tend to move in the same way as the iron, though not to so great an extent. If however we use bismuth instead of iron, we shall find that the bismuth is repelled from the magnet, instead of being attracted towards it; the bismuth tending to move from the strong to the weak parts of the field; the effect is however very small compared with that exhibited by iron; and to make the repulsion evident it is necessary to use a strong electromagnet. When the positive pole of a magnet is brought near a bar of bismuth the end of the bar next the positive pole becomes itself a positive pole, while the further end of the bar becomes a negative pole.

Substances which behave like iron, i.e. which move from the weak to the strong parts of the magnetic field, are called *paramagnetic* substances; while those which behave like bismuth, and tend to move from the strong to the weak parts of the field, are called *diamagnetic* substances.

When tested in very strong fields all substances are found to be para- or dia-magnetic to some degree, though the extent to which iron transcends all other substances is very remarkable.

153. Magnetic Force and Magnetic Induction. The magnetic force at any point in air is defined to be the force on unit pole placed at that point, or—what is equivalent to this—the couple on a magnet of unit moment placed with its axis at right angles to the magnetic force. When however we wish to measure the magnetic force inside a magnetizable substance, we have to make a cavity in the substance in which to place the magnet used in measuring the force. The walls of the cavity will however become magnetized by induction, and this magnetization will affect the force inside the cavity. The magnetic force thus depends upon the shape of the cavity, and this shape must be specified if the expression *magnetic force* is to have a definite meaning.

Let P be a point in a piece of iron or other magnetizable substance, and let us form about P a cylindrical cavity, the axis of the cylinder being parallel to the direction of magnetization at P. Let us first take the case when the cylinder is a very long and narrow one. Then in consequence of the magnetization at P, there will be a distribution of positive magnetism over one end of the cylinder, and a distribution of negative magnetism over the other. Let I be the intensity of the magnetization at P, reckoned positive when the axis of the magnet is drawn from left to right, then when the cylindrical cavity has been formed round P there will be, if α is the cross-section of the cavity, a charge $I\alpha$ of magnetism on the end to the left, and a charge $-I\alpha$ on the end to the right. If $2l$, the length of the cylinder, is very great compared with the diameter, then the force on unit pole at the middle of the cylinder due to the magnetism at the ends of the cylinder will be $2I\alpha/l^2$, and will be indefinitely small if the breadth of the cylinder is indefinitely small compared with its length. In this case the force on unit pole in the cavity is independent of the intensity of magnetization at P. The force in this cavity is defined to be '*the magnetic force at P.*' Let us denote it by H.

Let us now take another co-axial cylindrical cavity, but in this case make the length of the cylinder very small compared with its diameter, so that the shape of the cavity is that of a narrow crevasse.

he left end of this crevasse there is a charge of magnetism of ace density I, and on the right end of the crevasse a charge of netism of surface density $-I$. If a unit pole be placed inside crevasse the force on it due to this distribution of magnetism be the same as the force on unit charge of electricity placed veen two infinite plates charged with electricity of surface ity $+I$ and $-I$ respectively, i.e. by Art. 14, the force on the pole in this case will be $4\pi I$. Thus in a crevasse the total force he unit pole at P will be the resultant of *the magnetic force* at nd a magnetic force $4\pi I$ in the direction of the magnetization . The force on the unit pole in the crevasse is defined to be the *gnetic induction*' at P, we shall denote it by B. If we had taken vity of any other shape the force due to the magnetization at vould have been intermediate in value between zero for the long nder and $4\pi I$ for the crevasse; thus if the cavity had been spherical force due to the magnetization would (Art. 139) have been /3.

The magnetic induction is not necessarily in the same direction he magnetic force, it will only be so when the magnetization at parallel to the magnetic force.

154. Tubes of Magnetic Induction. A curve drawn such its tangent at any point is parallel to the magnetic induction at point is called a line of magnetic induction: in non-magnetizable stances the lines of magnetic induction coincide with the lines nagnetic force. We can also draw tubes of magnetic induction as we draw tubes of magnetic force.

We shall choose the unit tube so that the magnetic induction at place whether in the air or iron is equal to the number of tubes induction which cross a unit area at right angles to the uction.

Let us consider the case of a small bar magnet, the magnetism ng entirely at its ends. Suppose A and B are the ends of the gnet, A being the negative, B the positive end, then in the air lines of magnetic induction coincide with those of magnetic force go from B to A. To find the lines of magnetic induction at a nt P inside the magnet, imagine the magnet cut by a plane at nt angles to the axis and the two portions separated by a short

distance, the lines of magnetic force in this short air space w the lines of magnetic induction in the section through P. I magnet is cut as in the figure then the end C will be a positive of the same strength as A, the end D a negative pole of the strength as B. Thus through the short air space between C a tubes of induction will pass running in the direction AB. Dr closed surface passing through the between C and D and enclosing A DB. The magnetic force at any poir this surface is equal to the magnetic duction at the same point due to undivided magnet. Since this su encloses as much positive as nega magnetism, we see as in Art. 10 that total magnetic force over its su vanishes. Hence we see that the t of induction inside the magnet are e in number at each cross section and this number is the sam the number of those which leave the pole B and enter A. In the lines of magnetic induction due to the magnet form a serie closed curves all passing through the magnet and then sprea out in the air, the lines running from B to A in the air and fro to B in the magnet.

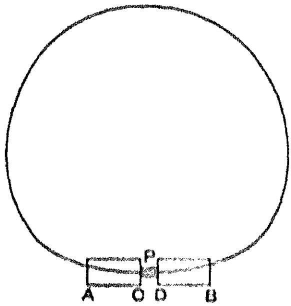


Fig. 80

Thus we may regard any small magnet, whose intensity is I area of cross section a, as the origin of a bundle of closed tube induction, the number of tubes being $4\pi Ia$; every tube in this bu passes through the magnet, running through the magnet in direction of the magnetization.

It is instructive to compare the differences between the proper of the tubes of electric polarization in electrostatics and those magnetic induction in magnetism: the most striking difference that whereas in electrostatics the tubes are not closed but begi positive electrification and end on negative, in magnetism the tu of induction always form closed curves and have neither beginn nor end.

A surface charged with electricity of surface density σ is origin of σ tubes of electric polarization per unit area. A sm magnet whose intensity of magnetization is I is the origin of 4

tubes of magnetic induction per unit area of cross-section of the magnet, all these tubes passing through the magnet which acts as a kind of girdle to them.

The properties of these tubes are well summed up by Faraday in the following passage (*Experimental Researches*, § 3117): "there exist lines of force within the magnet, of the same *nature* as those without. What is more, they are exactly equal in *amount* to those without. They have a relation in *direction* to those without and in fact are continuations of them, absolutely unchanged in their nature so far as the experimental test can be applied to them. Every line of force, therefore, at whatever distance it may be taken from the magnet, must be considered as a closed circuit passing in some part of its course through the magnet, and having an equal amount of force in every part of its course." Faraday's lines of force are what we have called tubes of induction.

155. We shall now proceed to consider the special case, including that of iron and all non-crystalline substances when magnetized entirely by induction, in which the direction of the magnetization and consequently of the magnetic induction is parallel to the magnetic force. Let H be the magnetic force, B the magnetic induction, and I the intensity of magnetization, then we have by Art. 153,

$$B = H + 4\pi I.$$

The ratio of I to H when the magnetization is entirely induced is called the *magnetic susceptibility* and is usually denoted by the letter k. The ratio of B to H under the same circumstances is called the *magnetic permeability* and is denoted by the letter μ.

We thus have $\qquad I = kH,$

$$B = \mu H,$$

and since $\qquad B = H + 4\pi I,$

we have $\qquad \mu = 1 + 4\pi k.$

The quantity μ which occurs in magnetism is analogous to the specific inductive capacity in electrostatics; but while as far as our knowledge at present goes, the specific inductive capacity at any time does not depend much, if at all, upon the value of the electric intensity at that time, nor on the electric intensity to which the

dielectric has previously been exposed; the permeability, on the other hand, if the magnetic force exceeds a certain value (about 1/10 of the earth's horizontal force), depends very greatly upon the magnitude of the magnetic force, and also upon the magnetic forces which have previously been applied to the iron. The variations in the magnetic permeability are most conveniently represented by

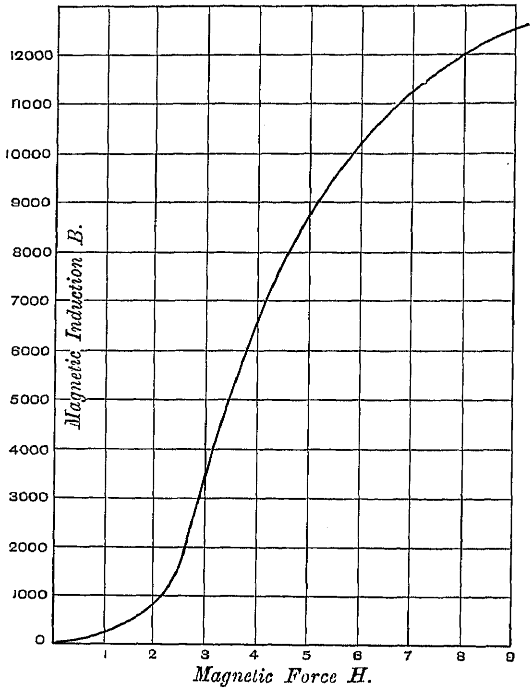


Fig. 81

curves in which the ordinate represents the magnetic induction, the abscissa the corresponding magnetic force. If P be a point on such a curve, PN the ordinate, ON the abscissa, then the magnetic permeability is PN/ON.

Such a curve is shown in Fig. 81, in which the ordinates represent for a particular specimen of iron the values of B, the magnetic induction, the abscissæ the values of H, the magnetic force. For small values of H the curve is straight, indicating that the permeability is independent of the magnetic force. When however the

ic force increases beyond about $\frac{1}{10}$ of the earth's horizontal r about ·018 in C.G.S. units, the curve begins to rise rapidly, e value of μ is greater than it was for small magnetic forces. rve rises rapidly for some time, the maximum value of μ ng when the magnetic force is about 5 C.G.S. units, then it to get flatter and there are indications that for very great of the magnetic force the curve again becomes a straight line

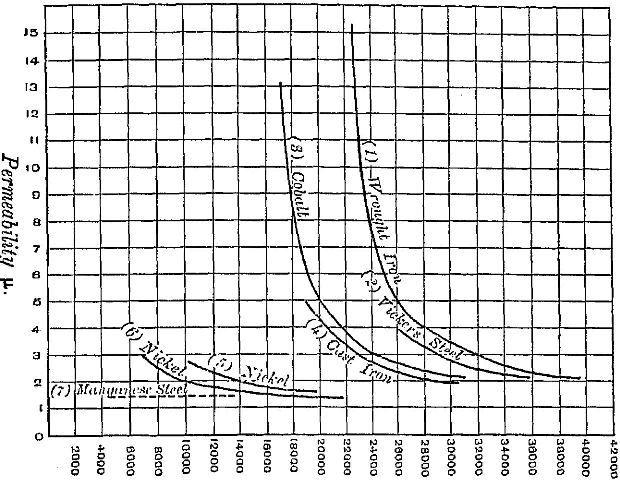


Fig. 82

g an angle of 45° with the axis along which the magnetic s measured. The relation between B and H along this part of rve is

$$B = H + \text{constant}:$$

ring this with the relation

$$B = H + 4\pi I,$$

e that it indicates that the intensity of magnetization has e constant. In other words, the intensity of magnetization ot increase as the magnetic force increases. When this is the ıe iron or other magnetizable substance is said to be 'saturated.' iron seems not to be able to be magnetized beyond a certain ity. In a specimen of soft iron examined by Prof. Ewing, tion was practically reached when the magnetic force was

about 2000 in C.G.S. units. For steel the magnetic force required for saturation is very much greater than for soft iron, and in some specimens of steel examined by Prof. Ewing saturation was not attained even when the magnetizing force was as great as 10,000. For a particular kind of steel called Hadfield's manganese steel the value of μ was practically constant even in the strongest magnetic fields, this steel however is only slightly magnetic, the value of μ being about 1·4. The greatest value of μ which has been observed is 20,000 for soft iron, in this case however the iron was tapped when under the influence of the magnetic force. Fig. 82 represents the results of Ewing's experiments on the relation between magnetic permeability and magnetic induction in very intense magnetic fields.

156. Effect of Temperature on the Magnetic Permeability. The permeability of iron depends very much upon the temperature. Dr J. Hopkinson found that as the temperature increases, starting from about 15° C., the magnetic permeability at first slowly increases; this slow rate of increase is however exchanged for an exceedingly rapid one when the temperature approaches a 'critical temperature' which for different samples of iron and steel ranges from 690° C. to 870° C., at this temperature the value of the permeability is many times greater than that at 15° C.: after passing this value the permeability falls even more rapidly than it previously rose. Indeed so fast is the fall that at a few degrees above the critical temperature iron practically ceases to be magnetic. Just below this temperature iron is an intensely magnetic substance, while above that temperature it is not magnetic at all. There are other indications that iron changes its character in passing through this temperature, for here its thermo-electric properties as well as its electrical resistance suffer abrupt changes. This temperature is often called the temperature of recalescence from the fact that a piece of iron wire heated above this temperature to redness and then allowed to cool, will get dull before reaching this temperature and will glow out brightly again when it passes through it.

Though the value of μ at higher temperatures (lower however than that of recalescence) is for small magnetic forces greater than at lower temperatures, still as it is found that at the higher tempera-

tures iron is much more easily saturated than at lower ones, the value of μ for the hot iron might be smaller than for the cold if the magnetic forces were large.

Hopkinson found that some alloys of nickel and iron after being rendered non-magnetic by being raised above the temperature of recalescence remained non-magnetic when cooled below this temperature; it was not until the temperature had fallen far below the temperature of recalescence that they regained their magnetic properties. Thus these alloys can at one and the same temperature exist in both the magnetic and non-magnetic states.

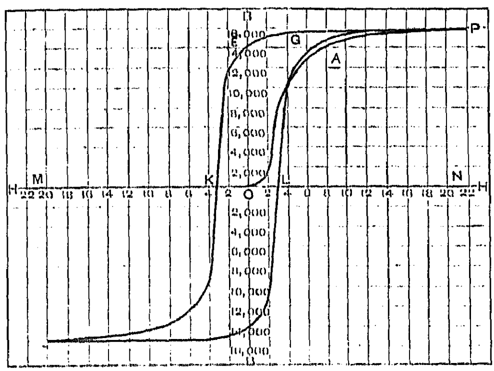


Fig. 83

157. Magnetic Retentiveness. Hysteresis. When a piece of iron or steel is magnetized in a strong magnetic field it will retain a considerable proportion of its magnetization even after the applied field has been removed and the iron is no longer under the influence of any applied magnetic force. This power of remaining magnetized after the magnetic force has been removed, is called magnetic retentiveness; permanent magnets are a familiar instance of this property. This effect of the previous magnetic history of a substance on its behaviour when exposed to given magnetic conditions has been studied in great detail by Prof. Ewing, who has given to this property the name of hysteresis. To illustrate this properly, let us consider the curve (Fig. 83) which is taken from Prof. Ewing's paper on the magnetic properties of iron (*Phil. Trans.* Part II., 1885), and which represents the relation for a sample of soft iron between the

intensity of magnetization (the ordinate) and the
(the abscissa), when the magnetic force increases
ON, then diminishes from *ON* through zero to
increases again up to its original value. When
applied we have the state represented by the p
curve, which begins by being straight, then increa
bends round and finally reaches *P*, the point cor
greatest magnetic force applied to the iron. If
diminished it will be found that the magnetization
is greater than it was when the magnet was initially
of the same force, i.e. the magnet has retained so
magnetization, thus the curve *PE*, when the for
will not correspond to the curve *OP* but will be ab
magnetization retained by the magnet when fre
force; in some cases it amounts to more than 90
greatest magnetization attained by the magnet. V
izing force is reversed the magnet rapidly loses
and the negative force represented by *OK* is suf
it of all magnetization. When the negative mag
creased beyond this value, the magnetization is ne
magnetic force is again reversed it requires a posit
OL to deprive the iron of its negative magnetiz
force is again increased to its original value the rel
force and induction is represented by the portion *I*
If after attaining this value the force is again dim
and back again, the corresponding curve is the cu

From the fact that this curve encloses a finite a
a certain amount of energy must be dissipated a
heat when the magnetic force goes through a co
show this let us suppose that we have a small magne

then taking this direction as the axis of x, the force
pole of the magnet is

$$\left(H + \frac{l}{2}\frac{dH}{dx}\right) I\alpha,$$

that on the negative pole is

$$-\left(H - \frac{l}{2}\frac{dH}{dx}\right) I\alpha.$$

Thus the force on the magnet is

$$l\alpha I \frac{dH}{dx},$$

the work done on the magnet when it moves through

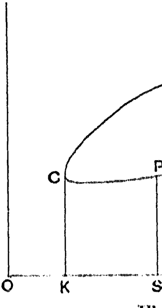
$$l\alpha\, I \frac{dH}{dx} \,.\, \delta x$$

or $$l\alpha I\delta H.$$

Since $l\alpha$ is the volume of the magnet the work done
per unit volume is $I\delta H$.

If in Fig. 84 $OS = H, OT = H + \delta H$ and $SP =$
represented on the diagram by the area $SPQT$.

Thus the total work done by the magnet when
place where the force is OK to one where it is OL is
the area $CKLDE$. Let the magnet now be pulled back from the place where the force is OL to the place from which it started where the force is OK, work has to be done on the magnet and this work is represented by the area $DFCKL$. Thus the excess of the work done on the magnet over that done by the magnet, when the magnetic force



Fi

Another proof of this is given in Cha

If instead of a curve showing the we use one showing the relation betwe similar loops in this second curve an will be 4π times the area of the corres H curve.

For the area of a loop on the first cu

$$-\int IdH,$$

this is equal to $$-\frac{1}{4\pi}\int (B-H)$$

$$= -\frac{1}{4\pi}\int BdH,$$

since $\int HdH = 0$, as the initial and final area of a loop on the B and H curve is

$$-\int BdH.$$

Hence we see that this area is 4π times t loop on the I and H curve.

157*. A good deal of light is thrown magnetic intensity and the magnetic for of temperature on this relation by cor whose molecules are little magnets. In field the directions of the axes of these distributed and the number of molecules angles between θ and $\theta + \delta\theta$ with a fix to $\sin\theta\, d\theta$. If however an external ma molecules it will tend to make their ax the magnetic force; it will not however direction because the collisions between t knock the axes out of line. We can c

mber of magnets whose axes make angles between θ and $\theta +$
h the direction of the magnetic force will be

$$C\,\epsilon^{\frac{HM\cos\theta}{RT}}\sin\theta\,.\,d\theta,$$

$$C\,\epsilon^{a\cos\theta}\sin\theta\,d\theta,$$

$$a = HM/RT;$$

moment of these magnets in the direction of the magnetic forc

$$CM\cos\theta\,\epsilon^{a\cos\theta}\sin\theta\,d\theta.$$

nce if N is the number of magnets per unit volume and I
gnetic moment of unit volume

$$N = C\int_0^{\pi}\epsilon^{a\cos\theta}\sin\theta\,.\,d\theta,$$

$$I = CM\int_0^{\pi}\cos\theta\epsilon^{a\cos\theta}\sin\theta\,d\theta;$$

nce $$I = NM\left(\frac{\epsilon^a+\epsilon^{-a}}{\epsilon^a-\epsilon^{-a}}-\frac{1}{a}\right) \quad\text{.....................(1}$$

The curve representing the relation between I and a is show
g. 84 a.

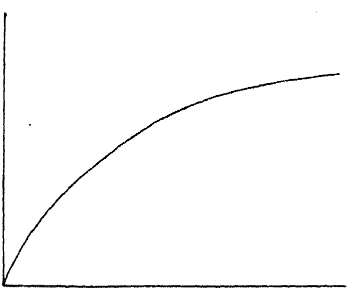


Fig. 84 a

Since $a = MH/RT$, a is proportional to H so that the cur
e I and H curve for the gas.

and is thus inversely proportional to Curie showed that this relation betw existed for a considerable number of

The preceding relation between I a hysteresis, this will however occur if acting on the molecules is made up of t the external field the other a field due molecules and proportional to I, where netization: let this part of the magneti the external field H_1, then if H is the t

$$H = H_1 + \lambda$$

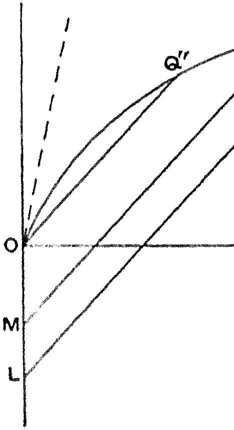


Fig. 84 *b*

and since $$H = a \cdot \frac{RT}{M}$$

we have $$I = \frac{RT}{\lambda M} a -$$

Hence to find the value of I correspo

Suppose Q (Fig. 84 b) is the point of intersecti
ternal field is $H_1 = OL$, let the field diminish to OM
magnetization will be the ordinate of Q' where M
LQ, let now the external field become zero, then t
be Q'' where OQ'' is parallel to LQ; hence in this c
magnetization when the force is removed, i.e. there
magnetization. The line corresponding to the zer
intersect the curve if like the dotted line it is steeper
to the curve at the origin. The equation to this tange

$$y = \tfrac{1}{3}NM \,.\, a,$$

hence for the tangent to be steeper than the line

$$\frac{1}{3}NM > \frac{RT}{\lambda M},$$

or

$$T < \frac{1}{3}\frac{NM^2\lambda}{R},$$

thus if T is greater than this value there will be n
netism, while if T is less there will; hence if

$$T_0 = \frac{1}{3}\frac{NM^2\lambda}{R},$$

T_0 will be a critical temperature for the magnetic p
substance.

For small values of the magnetic force

$$y = \tfrac{1}{3}NM \,.\, a = \frac{RT_0}{\lambda M}\, a,$$

and also

$$= \frac{RT}{\lambda M}\, a - \frac{H_1}{\lambda},$$

or

$$\frac{y\,(T - T_0)}{T_0} = \frac{H_1}{\lambda},$$

so that

$$k = \frac{y}{H_1} = \frac{T_0}{T - T_0} \cdot \frac{1}{\lambda}.$$

1. The magnetic force parallel to the surfac
in the two media.

2. The magnetic induction at right angles t
be the same in the two media.

To prove the first condition, let P and Q be
the surface of separation, Q being in the first
medium. Now the magnetic force at a point is
Art. 153) the force on a unit pole placed in a cavi
when the magnetism on the walls of the cavity
hence since this magnetism is to be disregar
between the magnetic forces at P and Q mu
magnetism on the surface between P and Q: bu
at right angles to this portion of the surface du
are different at P and Q, the forces parallel to
same. Hence we see that the tangential magneti
same at P as at Q.

We shall now show that the normal magneti
tinuous. All the tubes of magnetic induction f
Hence any tube must cut a closed surface an eve
half these times it will be entering the surface, h
contributions of each tube to the total normal mag
be the same in amount but opposite in sign when
it leaves the surface. Hence the total contribut
zero, and thus the total normal magnetic inducti
surface vanishes. Consider the surface of a very s
sides are parallel to the normal at P, one end b
(1), the other in (2). The total normal inductio
is zero, but as the area of the sides is negligible
of the ends, this implies that the total normal i

. When there is no free electricity on the surface the norm
ric polarization must be the same in both. That is, if F, F'
normal electric intensities in the media whose specific inducti
cities are respectively K and K',

$$KF = K'F'.$$

f we compare these conditions with those satisfied at
ndary of two media in the magnetic field and remember t
n the magnetization is induced, the magnetic induction is eq
times the magnetic force, we see that we have complete analo

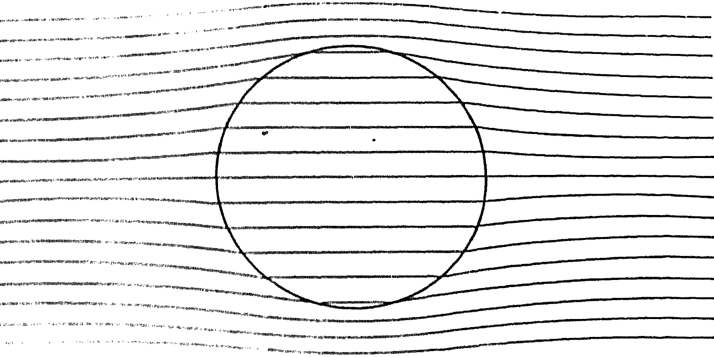


Fig. 85

ween the disturbance of an electric field produced by the prese
uncharged dielectrics and the disturbance in a magnetic
duced by para- or dia-magnetic bodies in which the magnet
ntirely induced.

Hence from the solution of any electrical problem we can ded
t of the corresponding magnetic one by writing magnetic f
electric intensity, and μ for K.

this case μ is greater than 1; when they go from a
substance they are bent towards the normal, sin
less than 1.

The effects produced when paramagnetic and d
are placed in a uniform field of force are shown i

159. If μ is infinite $\tan\theta_1$ vanishes, and the
in air are at right angles to the surface, so tha
substance of infinite permeability is a surface
potential. The surface of such a substance corresp
of an insulated conductor without charge in elec
problem relating to such conductors can be at c
corresponding case in magnetism. In particular
principle of images (Chap. v.) to find the effect
distribution of magnetic poles in presence of a
magnetic permeability.

160. Sphere in uniform field. We sh
that if a sphere, whose radius is a, and whose
capacity is K, is placed in a uniform electric fie
electric intensity before the introduction of th
field when the sphere is present will at a point P
consist of H and an electric intensity whose com
equal to

$$2H\frac{(K-1)}{K+2}\cos\theta\frac{a^3}{r^3},$$

and whose component at right angles to PO in th
to increase θ is

$$H\frac{(K-1)}{K+2}\sin\theta\frac{a^3}{r^3};$$

in these expressions $OP = r$, θ is the angle O

A very important special case is when μ is very large compa h unity. In this case the magnetic forces due to the sphere roximately

$$2H\frac{a^3}{r^3}\cos\theta$$

ng PO, and

$$H\frac{a^3}{r^3}\sin\theta$$

right angles to it.

Inside the sphere the magnetic force is

$$\frac{3}{\mu}H,$$

l is very small compared with that outside. The magn uction inside the sphere is $3H$. Thus through any area in nere at right angles to the magnetic force, three times as m es of induction pass as through an equal and parallel area at nite distance from the sphere.

The resultant magnetic force in air vanishes round the equa the sphere, and at any other point on the sphere is along rmal and equal to $3H\cos\theta$.

161. Magnetic Shielding. Just as a conductor is abl eld off the electric disturbance which one electrical system wo oduce on another, so masses of magnetizable material, for wh has a large value, will shield off from one system magnetic fo e to another. Inasmuch however as μ has a finite value for stances the magnetic shielding will not be so complete as ctrical.

162. Iron Shell. We shall consider the protection affor a spherical iron shell against a uniform magnetic field. We

Let a be the radius of the inner surface o
the outer surface. Let H be the force in the un
shell was introduced. Let the magnetic forces d
on the outer surface of the shell consist, at a
sphere, of a radial force

$$\frac{2M_1 \cos\theta}{r^3},$$

a tangential force

$$\frac{M_1 \sin\theta}{r^3},$$

where $r = OP$ and θ is the angle OP makes wit
The magnetic force due to this distribution o
uniform inside the sphere whose radius is b, it w
of H and be equal to $-M_1/b^3$.

Let the magnetization on the inner surface
to magnetic forces given by similar expressions
M_1 and a for b.

This system of forces, whatever be the va
satisfies the condition that as we cross the sur
tangential components of the magnetic force
must now see if we can choose M_1, M_2 so as
magnetic induction continuous.

The normal magnetic induction (reckoned
outward drawn normal) in the air just outsi
equal to

$$H \cos\theta + \frac{2M_1}{b^3}\cos\theta + \frac{2M_2}{b^3}\text{co}$$

the normal magnetic induction in the iron j
surface of the shell is equal to

The normal magnetic induction in the iron just outside the inn face of the shell is

$$\mu\left(H\cos\theta - \frac{M_1}{b^3}\cos\theta + \frac{2M_2}{a^3}\cos\theta\right),$$

normal magnetic induction in the air just inside the shell ual to

$$H\cos\theta - \frac{M_1}{b^3}\cos\theta - \frac{M_2}{a^3}\cos\theta;$$

ese are equal if

$$(\mu-1)\frac{M_1}{b^3} - (2\mu+1)\frac{M_2}{a^3} = (\mu-1)H \quad \text{.........(2).}$$

Equations (1) and (2) are satisfied if

$$M_1 = (\mu-1)H\,\frac{\{b^3(2\mu+1) - 2a^3(\mu-1)\}}{(2\mu+1)(\mu+2) - 2(\mu-1)^2\dfrac{a^3}{b^3}},$$

$$M_2 = -(\mu-1)H\,\frac{3a^3}{(2\mu+1)(\mu+2) - 2(\mu-1)^2\dfrac{a^3}{b^3}}.$$

The magnetic force in the hollow cavity is equal to

$$H - \frac{M_1}{b^3} - \frac{M_2}{a^3}.$$

Substituting the values of M_1 and M_2 we see that this ual to

$$\frac{9\mu H}{9\mu + 2(\mu-1)^2\left(1 - \dfrac{a^3}{b^3}\right)}.$$

If μ is very large compared with unity this is approximat ual to

$$\frac{H}{1 + \frac{2}{9}\mu\left(1 - \dfrac{a^3}{b^3}\right)}.$$

In the cases where $\mu = 1000$ and $\mu = 100$, t
force inside the sphere, to H for different values
the following table.

a/b	H'/H $\mu=1000$	H'/H $\mu=100$
·99	3/23	9/15
·9	1/67	1/7
·8	1/109	1/12
·7	1/146	1/15
·6	1/175	1/18
·5	1/195	1/20
·4	1/209	1/22
·3	1/216	1/22
·2	1/221	1/23
·1	1/223	1/23
·0	1/223	1/23

Galvanometers which have to be used in pla
action of extraneous magnets are sometimes protec
them with a thick-walled tube made of soft iron.

We may regard the shielding effect of the she
the tendency of the tubes of magnetic induction
possible through iron; to do this they leave the
into the shell.

163. Expression for the energy in the
We shall suppose that the field contains permane
as pieces of iron magnetized by induction. When
the permanent magnets is given, the magnetic f
minate. We may regard the lines of magnetic f
starting from the permanent magnets their dist

o the magnetic potential at that pole. Thus if Q is the poten
rgy of the magnetic field

$$Q = \tfrac{1}{2}\Sigma m\Omega,$$

ere m is the strength of a permanent pole and Ω the magn
ential at that pole. We may regard the pole as the origin
unit tubes of magnetic force. These will have similar proper
the tubes of electric force; we can show just as we did in Art.
t in air the magnetic force is 4π times the number of unit tu
ough unit area, while the equation $B = \mu H$ indicates that thro
medium of permeability μ the force is $\frac{4\pi}{\mu}$ times the number
es, or if N be the number of tubes $N = \frac{\mu H}{4\pi}$.

Hence by exactly the same method as in Art. 73 we can sh
t the energy of the system is the same as if it were distribu
ough the field in such a way that at a place where the magn
e is H the energy per unit volume is equal to

$$\frac{\mu H^2}{8\pi}.$$

164. When a tube of induction enters a paramagnetic substa
m air the resultant magnetic force is (when the magnetizatio
irely induced) less in the paramagnetic substance than in
energy per unit length will be less in the magnetic substa
n in the air since the energy per unit length of a tube of induct
proportional to the resultant magnetic force along it. Thus
ordance with the principle that when a system is in equilibr
potential energy is a minimum, the tubes of induction will t
leave the air and crowd into the magnet, when this act does

A small piece of iron placed in a magnetic fie
is not uniform will tend to move from the weak t
of the field, since by doing so it encloses a greate
of induction and thus produces a greater decre
The direction of the force tending to move the iro
along which the rate of increase of R^2 is greate
general the direction of the magnetic force. Th
a bar magnet AB, the greatest rate of increase i
equidistant from A and B is along the perpendi
C on AB, and this is the direction in which a s
at C will tend to move; it is however at right ang
of the magnetic force at C.

There will be no force tending to move a piece
in a uniform magnetic field.

A diamagnetic substance will tend to move
the weak parts of the field, since by so doing i
number of tubes of magnetic induction enclose
also the energy, for the tubes of induction hav
unit length when they are in the diamagnetic su
they are in air.

165. Ellipsoids. We have hitherto onl
case of spheres placed in a uniform field. Bodie
longer in one direction than another have very in
which are conveniently studied by investigatin
ellipsoids placed in a uniform magnetic field.

We saw in Art. 139 that the magnetic fiel
uniformly magnetized in the direction of the a
regarded as due to two spheres, one of uniform
centre at O', the other of uniform density $-\rho$ w

zation, it will coincide with the positive body if $\rho\xi = A$, A be intensity of magnetization of the body.

Let us suppose that the body is uniformly magnetized w ensity A in the direction of the axis of x, and let $\rho\Omega$ be ential of the positive body at the point P, then the potential negative body at P will be equal to $-\rho\Omega'$, where $\rho\Omega'$ is ential of the positive body at P', if PP' is parallel to the axis nd equal to ξ.

But since $P'P$ is small,

$$\rho\Omega' = \rho\left(\Omega + \xi\frac{d\Omega}{dx}\right).$$

potential of the negative body is therefore

$$-\rho\left(\Omega + \xi\frac{d\Omega}{dx}\right).$$

us the potential of the positive and negative bodies together, refore of the magnetized body, will be

$$-\rho\xi\frac{d\Omega}{dx},$$

$$= -A\frac{d\Omega}{dx},$$

ce $$\rho\xi = A.$$

If the body instead of being magnetized parallel to x is unifor gnetized so that the components of the intensity parallel to x, respectively A, B, C, the magnetic potential is

$$-\left(A\frac{d\Omega}{dx} + B\frac{d\Omega}{dy} + C\frac{d\Omega}{dz}\right) \quad \text{................(1}$$

We shall now show that if an ellipsoid is placed in a unif gnetic field it will be uniformly magnetized by induction. ove this it will be sufficient to show that if we superpose on to

attraction of an ellipsoid of uniform unit densit
Analytical Statics, vol. II. p. 112, equal to

$$Lx, \quad My, \quad Nz$$

respectively, where L, M, N are constant as long
coordinates are x, y, z is inside the ellipsoid.

Hence by (1) since

$$\frac{d\Omega}{dx} = -Lx, \text{ \&c.},$$

the magnetic potential inside the ellipsoid due to
will be

$$(ALx + BMy + CNz),$$

so that the magnetic forces parallel to the axes o
magnetization of the ellipsoid will be

$$-AL, \quad -BM, \quad -CN$$

respectively.

Hence if N_1 is the component of these forces
drawn normal to the surface of the ellipsoid,

$$N_1 = -(ALl + BMm + CNn)$$

where l, m, n are the direction cosines of the outw
If N_2 is the force due to the magnetization on
same direction just outside the ellipsoid, then

$$N_2 = N_1 + 4\pi(lA + mB + nC)$$

$$= lA(4\pi - L) + mB(4\pi - M) + nC$$

Let X, Y, Z be the components of the force
field. Then N_1', the total force inside the ellipsoi
drawn normal, will be given by the equation

$$N_1' = lX + mY + nZ + N_1,$$

and if N_2' is the total force just outside the ellip
ward drawn normal

But this condition will be satisfied if

$$\left.\begin{aligned} A &= \frac{(\mu-1)X}{4\pi + L(\mu-1)} \\ B &= \frac{(\mu-1)Y}{4\pi + M(\mu-1)} \\ C &= \frac{(\mu-1)Z}{4\pi + N(\mu-1)} \end{aligned}\right\} \quad \text{.....................(2}$$

These equations give the intensity of magnetization of an ellip aced in a uniform magnetic field.

The force inside the ellipsoid due to its magnetization has – BM, $-CN$ for components parallel to the axes of x, y, z res vely; these components act in the opposite direction to the exte ld and the force of which these are the components is called magnetizing force. We see from equations (2) that the compon the demagnetizing force are

$$-\frac{(\mu-1)LX}{4\pi + L(\mu-1)},$$

$$-\frac{(\mu-1)MY}{4\pi + M(\mu-1)},$$

$$-\frac{(\mu-1)NZ}{4\pi + N(\mu-1)}.$$

We shall now consider some special cases in detail. Let us e case of an infinitely long elliptic cylinder, let the infinite ax rallel to z, let $2a$, $2b$ be the axes in the direction of x and y; outh's *Analytical Statics*, vol. II. p. 112)

$$L = 4\pi\frac{b}{a+b}, \quad M = 4\pi\frac{a}{a+b}, \quad N = 0.$$

We see from this equation that A/X is appro when $(\mu - 1)\, b/(a + b)$ is very small, but only the way of measuring k is to measure A/X in the ca solid, magnetized along the long axis; but we see t elongated cylinder this will be equal to k only wh is very small. Now for some kinds of iron μ is hence if this method were to give in this case res per cent., the long axis would have to be 100,00 the short one. This extreme case will show the i very elongated figures when experimenting with permeability. Unless this precaution is taken the determine the value of a/b and not any magnetic pr

When the body is an elongated ellipsoid of r of the long to the short axis need not be so enorm of the cylinder, but it must still be very conside of x is the axis of revolution, then (Routh's *Analyt* p. 112) we have approximately

$$L = 4\pi \frac{b^2}{a^2}\left\{\log \frac{2a}{b} - 1\right\}, \quad M = N$$

Thus
$$\frac{A}{X} = \frac{k}{1 + (\mu - 1)\frac{b^2}{a^2}\left\{\log\frac{2a}{b} - 1\right\}}.$$

Thus if μ were 1000, the ratio of a to b would 900 to 1 in order that the assumption $A/X = k$ sh one per cent.

166. Couple acting on the Ellipsoid. the couple tending to twist the ellipsoid round th direction from x to y, is equal to

(volume of ellipsoid) $(YA - XB$

$a > b$, M is greater than L. Thus the couple tends to make g axis coincide in direction with the external force, so that psoid, if free to turn, will set with its long axis in the direct the external force. This will be the case whether μ is greate s than unity, i.e. whether the substance is paramagnetic or gnetic, so that in a uniform field both paramagnetic and gnetic needles point along the lines of force. It generally happ at a diamagnetic substance places itself athwart the lines gnetic force, this is due to the want of uniformity in the field nsequence of which the diamagnetic substance tries to get as m itself as possible in the weakest part of the field. This tende ries as $(\mu - 1)$; the couple we are investigating in this article va $(\mu - 1)^2$, and as $(\mu - 1)$ is exceedingly small for bismuth, uple will be overpowered unless the field is exceptionally unifo

167. Ellipsoid in Electric Field. The investigation . 165 enables us to find the distribution of electrification indu a conducting ellipsoid when placed in a uniform electric fi do this we must make μ infinite in the expressions of Art. 1 e quantity $lA + mB + nC$ which occurs in the magnetic probl responds to σ, the surface density of the electrification. Putt $= \infty$ in equations (2) we find

$$\sigma = \left\{\frac{lX}{L} + \frac{mY}{M} + \frac{nZ}{N}\right\}.$$

he force in the electric field is parallel to the axis of x

$$\sigma = \frac{lX}{L}.$$

us when the electric field is parallel to one of the axes of psoid, the density of the electrification is, as in the case o here, proportional to the cosine of the angle which the norma

If the ellipsoid is a very elongated one with its longer axis in t
rection of the electric force, then by Art. 165

$$\frac{4\pi}{L} = \frac{a^2}{b^2\left[\log\frac{2a}{b} - 1\right]}.$$

ius, when a/b is large, $4\pi/L$ is a large quantity, and the elect
tensity at the end of the ellipsoid is very large compared with t
tensity in the undisturbed field. Thus if $a/b = 100$, the elect
tensity at the end is about 2500 times that in the undisturb
ld. This result explains the power of sharply pointed conduct
discharging an electric field, for when these are placed in ev
moderate field the electric intensity at the surface of the conduc
great enough to overcome the insulating power of the air,
t. 37, and the electrification escapes.

If an ellipsoidal conductor is placed in a uniform field of for
right angles to the axis c and making an angle θ with the a
we see from § 166 that the couple round the axis of c tending
ake the axis of a move towards the external force is equal to

$$\frac{2\pi}{3} F^2 abc \frac{(M - L)}{LM} \sin 2\theta$$

hen F is the external electric force.

When the ellipsoid is one of revolution round the axis of a, a
is large compared with b, the couple is approximately

$$\frac{1}{6} \frac{a^3 F^2 \sin 2\theta}{\left(\log\frac{2a}{b} - 1\right)}.$$

CHAPTER IX

ELECTRIC CURRENTS

168. Let two conductors A and B be at different potentia being at the higher potential and having a charge of positi ctricity, while B is at a lower potential and has a charge gative electricity; then if A is connected to B by a metallic w potential of A will begin to diminish and A will lose some of sitive charge, the potential of B will increase and B will lose so its negative charge, so that in a short time the potentials of A a will be equalized. During the time in which the potentials and B are changing the following phenomena will occur: the w nnecting A and B will be heated and a magnetic field will oduced which is most intense near the wire. If A and B erely charged conductors, their potentials are equalized so rapid d the thermal and magnetic effects are in consequence so transie at it is somewhat difficult to observe them. If, however, aintain A and B at constant potentials by connecting them w e terminals of a voltaic battery the thermal and magnetic effe ll persist as long as the connection with the battery is maintain d are then easily observed.

The wire connecting the two bodies A and B at different potent said to be conveying a current of electricity, and when A is los positive charge and B its negative charge the current is said w from A to B along the wire.

Let us consider the behaviour of the Faraday tubes during

was one in which there was equilibrium betwe
the tubes and the lateral repulsion they exert
after the tubes in the wire have shrunk the l
exerted is annulled and there will therefor
pressure tending to push the surrounding tu
into the wire, where they will shrink like th
wire. This process will go on until all the t
stretched from *A* to *B* have been forced int
effects annulled.

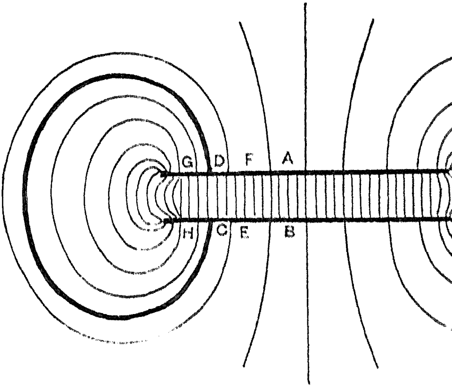


Fig. 86

The discharge of the conductors is thus
movement of the tubes in towards the wire
ends of these tubes along the wire. The posit
move on the whole from *A* towards *B* along t
ends from *B* towards *A*.

169. Strength of the current. If w

170. Electrodes. Anode. Cathode. If the ends R,
a body through which a current is flowing are portions of eq
otential surfaces, then R and S are called the electrodes, and if t
urrent is in the direction RS, R is called the anode and S the cathod

171. Electrolysis. In addition to the thermal and magne
fects mentioned in Art. 168, there is another effect characteris
the passages of the current through a large class of substanc
lled electrolytes. Suppose for example that a current pass
etween platinum plates immersed in a dilute solution of sulphu
id, then the solution suffers chemical decomposition to some exte
nd oxygen is liberated at the platinum anode, hydrogen at t
atinum cathode. There is no liberation of hydrogen or oxygen
e portions of the liquid not in contact with the platinum plat
owever far apart these plates may be. Substances whose co
ituents are separated in this way by the current are called *electrolyt*
nd the act of separation is called *electrolysis*. Electrolytes may
lids, liquids, or, as recent experiments have shown, gases. Iodi
silver is an example of a solid electrolyte, while as examples
quid electrolytes we have solutions of a great number of mine
lts or acids as well as many fused salts.

The constituents into which the electrolyte is separated by t
urrent are called the *ions*: the constituent which is deposited
e anode is called the *anion*, that which is deposited at the catho
e *cation*. With very few exceptions, an element, or such a gro
elements as is called by chemists a 'radical,' is deposited at t
me electrode from whatever compound it is liberated; thus
xample hydrogen and the metals are cations from whatever co
ounds they are liberated, while chlorine is always an anion.

The amount of the ions deposited by the passage of a curre
rough an electrolyte was shown by Faraday to be connected

passes as a very intense current for a short time or
current for a long time.

173. Faraday's Second Law of Elect
same quantity of electricity passes through differen
weights of the different ions deposited will be proportion
equivalents of the ions.

Thus, if the same current passes through a serie
from which it deposits as ions, hydrogen, oxygen, silv
then for every gramme of hydrogen deposited, 8 gra
108 grammes of silver and 35·5 grammes of chlorine v

If we define the *electro-chemical equivalent* of a
number of grammes of that substance deposited du
of the unit charge of electricity, we see that Faraday
comprised in the statement that the number of gra
deposited during the passage of a current through
equal to the number of units of electricity which have
the electrolyte multiplied by the electro-chemical e
ion, and that the electro-chemical equivalent is pro
chemical equivalent.

Elements which form two series of salts, such a
forms cuprous and cupric salts, or iron, which fo
ferric salts, have different electro-chemical equivale
they are deposited from solutions of the cuprous o
or ferric salts. The electro-chemical equivalents of a
are given in the following table; the numbers repre
in grammes of the substance deposited by the passag
magnetic unit of electricity (see Chap. XII.).

Hydrogen	·00010
Oxygen	·00082
Chlorine	·00367

rent. Imagine a plane drawn across the electrolyte, there m s in any time towards the cathode across the plane an amo the cation chemically equivalent to that of the anion deposi the same time at the anode; while a corresponding amount of on must cross the plane towards the anode. Thus in every p he electrolyte the cation is moving in the direction of the curre anion in the opposite direction.

Faraday's laws of electrolysis give a method of measuring antity of electricity which has passed through a conductor in e and hence of measuring the average current. For if we p electrolyte in circuit with the conductor in such a way that rent through the electrolyte is always equal to that through ductor, then the amount of the electrolyte decomposed will portional to the quantity of electricity which has passed thro conductor; if we divide the weight in grammes of the deposi of the ions by the electro-chemical equivalent of that ion the number of electro-magnetic units of electricity which sed through the conductor, dividing this by the time we get erage current in electro-magnetic units.

An electrolytic cell used in this way is called a voltameter; ms most frequently used are those in which we weigh the amo copper deposited from a solution of copper sulphate, or of si m a solution of silver nitrate, or measure the amount of hydro rated by the passage of the current through acidulated wate

174. Relation between Electromotive Force a rrent. Ohm's Law. The work done by the electric fo unit charge of electricity in going from a point A to another p s called the *electromotive force* from A to B. It is frequently writ the E.M.F. from A to B.

Ohm's Law is that the quantity R defined by
pendent of the strength of the current flowi
and depends only upon the shape and size of t
of which it is made, and upon its temperature

The most searching investigations have bee
of this law when currents pass through metals
have all failed to discover any exceptions to
accuracy with which resistances can be measur
gations an accuracy of one part in 100,000 h
tests to which it has been subjected are except

Ohm's Law does not however hold when the
rarefied gases.

175. Resistance of a number of Co
Suppose we have a number of wires AB, CD,

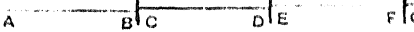


Fig. 87

nected together so that B is in contact with C, D
so on. This method of connection is called putt

Let r_1, r_2, r_3 ... be the resistances of the
and let i be the current entering the circuit
the current i will flow through each of the
consider the case when the field is steady, th
denote the potentials at A, B, C, &c. respecti
A to B is $v_A - v_B$; thus we have by Ohm's

$$v_A - v_B = r_1 i,$$
$$v_C - v_D = r_2 i,$$
$$v_E - v_F = r_3 i,$$

mparing this expression with the preceding, we see that

$$R = r_1 + r_2 + r_3 + \dots$$

nce when a system of conductors are put in series, the resista the series is equal to the sum of the resistances of the individ ductors.

176. Resistance of a number of Conductors arran Parallel. If the wires instead of being arranged so that l of one coincides with the beginning of the next, as in the mple, are arranged as in Fig. 88, the beginnings of all the w ng in contact, as are also their ends, the resistances are said arranged in parallel, or in multiple arc.

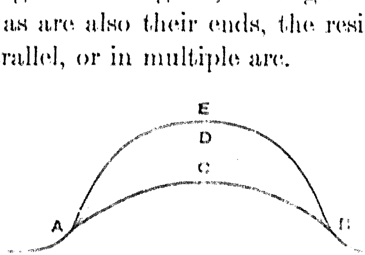


Fig. 88

We proceed now to find the resistance of a system of wire anged. Let i be the current flowing up to A, let this divide it currents i_1, i_2, i_3 ... flowing through the circuits ACB, A EB ... whose resistances are r_1, r_2, r_3 ... respectively. The v_B are the potentials of A and B respectively, we have by Oh w

$$v_A - v_B = r_1 i_1,$$

$$v_A - v_B = r_2 i_2,$$

$$v_A - v_B = r_3 i_3,$$

....

Now

or the reciprocal of the resistance of a number
parallel is equal to the sum of the reciprocals
resistances. The reciprocal of the resistance of a
its *conductivity*, hence we see that we may expres
investigation by saying that the conductivity of
ductors in parallel is equal to the sum of the co
individual conductors.

In the special case when all the wires connec
arc have the same resistance, and if there are n wi
when in multiple arc is $1/n$ of the resistance of on
wires.

177. Specific resistance of a substanc
wire whose length is l and whose cross section
area a, we may regard it as built up of cubes
unit length, in the following way; take a filament
l of these cubes in series and then place a of
parallel; the resistance of this system is evidentl
of the wire under consideration. If σ is the resist
cubes the resistance of the filament formed by pl
in series is $l\sigma$, and when a of these filaments are
the resistance of the system is $l\sigma/a$; hence the resis

$$\frac{l\sigma}{a}.$$

Since σ only depends on the material of which th
see that the resistance of a wire of uniform cr
portional to the length and inversely proportional
cross section.

The quantity denoted by σ in the preceding e
the specific resistance of the substance of which

t positive charge when it goes from A to B, and on unit negat rge when it goes from B to A; hence if in unit time N units itive charge go from A to B and N' units of negative charge fr o A, the work done is $E(N+N')$. But $N+N'$ is equal to strength of the current flowing from A to B, thus the work d qual to EC. If R is the resistance of the conductor betweer l B, $E=RC$; thus the work done in unit time is equal to R see that the same amount of work would be spent in driv urrent of the same intensity in the reverse direction, viz. fr o A.

By the principle of the Conservation of Energy the work sp the electric forces in driving the current cannot be lost, it n e rise to an equivalent amount of energy of some kind or ot e passage of the current heats the conductor, but if the hea used to leave the conductor as soon as produced the state of nductor is not altered by the passage of the current. echanical equivalent of the heat produced in the conductor own by Joule to be equal to the work spent in driving the curr rough the conductor, so that the work done in driving the curr n this case entirely converted into heat. Thus if H is the mechan uivalent of the heat produced in time t,

$$H = RC^2t.$$

e law expressed by this equation is called Joule's Law. It st at the heat produced in a given time is proportional to the squ the strength of the current.

Since by Ohm's Law $E=RC$, the heat produced in the t s also equal to

$$\frac{E^2}{R}t = ECt.$$

total amount of heat produced by the battery cell (α) must ivalent to that developed by the combination of the amount consumed in the cell while the current is passing with ivalent amount of sulphuric acid.

180. Electromotive Force of a Cell. If C is the curre he resistance of the wire between the plates, r that of the liq ween the plates, t the time the current has been flowing, t Joule's law the mechanical equivalent of the heat generated wire is RC^2t, that of the heat generated in the liquid is r shall see in Chapter XIII, that when a current flows across ction of two different metals, heat is produced or absorbed junction; this effect is called the Peltier effect. The la verning the thermal effects at the junction of two metals di y materially from Joule's law. The heat developed in accorda h Joule's law in a conductor AB is, as long as the strength current remains unaltered, the same whether the current fl m A to B or from B to A. The thermal effects at the junctio metals C and D depend upon the direction of the current; t here is a development of heat when the current flows across nction from C to D, there will be an absorption of heat at nction, if the current flows from D to C. These heat effects wl ange sign with the current are called reversible heat effects. at developed at the junction of two substances in unit tim ectly proportional to the strength of the current and not to uare of the strength.

In the case of the voltaic cell formed of dilute acid and zinc pper plates, the current passes across the junctions of the d acid and of the acid and copper as well as across the met

have been converted into zinc sulphate. Let w b
equivalent of the heat produced when one gramme
into zinc sulphate, then the mechanical equivalent
would be developed by the chemical action which
in the cell is $eCtw$; but this must be equal to the mech
of the heat developed in the cell, and hence we ha

$$RC^2t + rC^2t + PCt = eCtw,$$

or

$$(R + r)\,C = ew - P.$$

The quantity on the right-hand side is called the el
of the cell.

We see that it is equal to the sum of the produ
through the external circuit and the external re
current through the battery and the battery resi

We shall now prove that if the zinc and copper
being joined by a wire are connected to the plate
then if these plates are made of the same materia
different potentials, and the difference between the
equal the electromotive force of the battery. For
has got into a state of equilibrium, and any chang
electrical conditions, the increase in the electrical en
the energy lost in making the change. Suppose th
of the plate of the condenser in connection with th
the battery exceeds by E the potential of the ot
condenser in connection with the zinc plate of t
suppose now that the electrical state is altered b
electricity equal to δQ passing from the plate of t
low potential to the plate at high potential thro
from the zinc to the copper. The electrical energy
is increased by $E\delta Q$, while the passage of this quant

$Q \times w$, and this must be equal to the heat produced plus
rease in the electrical energy, hence we have

$$E\delta Q + P\delta Q = e\delta Q \times w,$$

$$E = ew - P,$$

t is, the difference of potential between the plates of the conde
equal to the electromotive force of the battery. Hence we
ermine this electromotive force by measuring the difference
ential.

The simple form of voltaic cell just described does not giv
nstant electromotive force, as the hydrogen produced by
emical action does not all escape from the cell; some of it adh
the copper plate, forming a gaseous film which increases
istance and diminishes the electromotive force of the cell.

The copper plate with the hydrogen adhering to it is said to
arized and to be the seat of a back electromotive force wh
kes the electromotive force of the battery less than its maxim
oretical value. We shall perhaps get a clearer view of the c
ion of the copper plate with its film of hydrogen from the follow
nsiderations. The hydrogen in an electrolyte follows the curr
l thus behaves as if it had a positive charge of electricity; if n
hydrogen ions when they come up to the copper plate, do
once give up their charges to the plate, but remain charged
mall distance from it; we shall have what is equivalent t
rged parallel plate condenser at the copper plate, the positiv
rged hydrogen atoms corresponding to the positive plate of
denser, and the copper to the negative plate. If the positiv
rged hydrogen ions charge up the positive plate of this conden
ving off by induction an equal positive charge from the cop
te, instead of giving up their charge directly to this plate;

Another cause of inconstancy is that the zinc
acts as an electrolyte and carries some of the c
travelling with the current, is deposited against
and alters the electromotive force of the cell.

The deposition of hydrogen against the posi
battery, and its liberation as free hydrogen, can be
ways; in the Bichromate Battery the copper pla
carbon, and potassium bichromate is added to th
as the bichromate is an active oxidising agent it oxid
as soon as it is formed, and thus prevents its acc
positive plate.

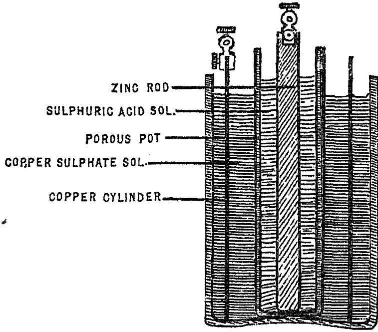


Fig. 89

181. Daniell's Cell. In Daniell's cell, t
phuric acid are enclosed in a porous pot (Fig. 89)
earthenware; the copper electrode usually takes
cylindrical copper vessel, in which the porous po
space between the porous pot and the copper is fille
solution of copper sulphate, in which crystals of co

nge takes place in the condition of the positive pole and there polarization.

182. Calculation of E.M.F. of Daniell's Cell. T emical energy lost in the cell during the passage of one unit ctricity may be calculated as follows: in the porous pot we ha e electro-chemical equivalent of zinc sulphate formed while o uivalent of sulphuric acid disappears; in the fluid outside this p e equivalent of sulphuric acid is formed and one equivalent pper sulphate disappears, thus the chemical energy lost is th ich is lost when the copper in one electro-chemical equivale copper sulphate is replaced by the equivalent quantity nc.

Now the electro-chemical equivalent of copper is ·003261 gramm d when 1 gramme of copper is dissolved in sulphuric acid the he ven out is 909·5 thermal units, or $909{\cdot}5 \times 4{\cdot}2 \times 10^7$ mechani its, since the mechanical equivalent of heat on the C.G.S. syst $4{\cdot}2 \times 10^7$. Thus the heat given out when one electro-chemi uivalent of copper is dissolved in sulphuric acid is

$$\cdot 003261 \times 909{\cdot}5 \times 4{\cdot}2 \times 10^7 = 1{\cdot}245 \times 10^8$$

echanical units.

The electro-chemical equivalent of zinc is ·003364 grammes, a e heat developed when 1 gramme of zinc is dissolved in sulphu id is $1670 \times 4{\cdot}2 \times 10^7$ mechanical units. Hence the heat develo hen one electro-chemical equivalent of zinc is dissolved in sulph id is $\cdot 003364 \times 1670 \times 4{\cdot}2 \times 10^7 = 2{\cdot}359 \times 10^8$ mechanical uni

Thus the loss of chemical energy in the porous pot is $2{\cdot}359 \times$ hile the gain in the copper sulphate is $1{\cdot}245 \times 10^8$, thus the t ss is $1{\cdot}114 \times 10^8$. Thus *ew* in Art. 180 $= 1{\cdot}114 \times 10^8$. The elec otive force of a Daniell's cell is about $1{\cdot}028 \times 10^8$. We see f

acid are placed in a porous pot, and this is placed in
glazed earthenware containing nitric acid; the positiv
of platinum foil dipping into the nitric acid. This
electromotive force, viz. $1{\cdot}97 \times 10^8$.

Bunsen's cell is a modification of Grove's,
platinum is replaced by hard gas carbon.

184. Clark's cell, which on account of it
very useful as a standard of electromotive force, is r

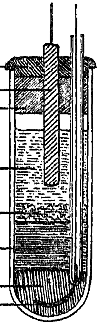


Fig. 90

The outer vessel (Fig. 90)
tube containing a glass tu
a platinum wire passes;
pure redistilled mercury su
the end of this wire is th
the tube; on the mercur
made by mixing mercurous
rated zinc sulphate and a
to neutralize it; a rod of
into the paste and is held
passing through a cork in
the test-tube. The electromotive force of this cell is
15° Centigrade.

Cadmium cell. In this cell the zinc of th
replaced by Cadmium, the negative electrode instea
is an amalgam containing twelve parts by weight
100 of the amalgam; the zinc sulphate solution is
saturated solution of Cadmium sulphate; the rest o
same as in the Clark cell. This cell has a smaller temper
than the Clark cell and is the one now most frequ
standard; its electromotive force at t° C. is

current enters the cell, while hydrogen adheres to the ot te B, by which the current leaves the cell. If these plates are n connected from the battery and connected by a wire, a curr l flow round the circuit so formed, the current going from te B to the plate A through the electrolyte and from A to ough the wire. This current is thus in the opposite direction t which originally passed through the cell. The plates are s be polarized, and the E.M.F. round the circuit, when they are f nected by the wire, is called the electromotive force of polari n. When the plates are disconnected from the battery and c cted by the wire the hydrogen and oxygen gradually disapp m the plates as the current passes. In fact we may regard arized plates as forming a voltaic battery, in which the chem ion maintaining the current is the combination of hydrogen ygen to form water. Though hydrogen and oxygen do not comb ordinary temperatures if merely mixed together, yet the oxy l hydrogen condensed on the platinum plates combine readily n as these plates are connected by a wire so as to make ygen and hydrogen parts of a closed electrical circuit. There nerous other examples of the way in which the formation of s ircuit facilitates chemical combination.

186. A Finite Electromotive Force is required erate the Ions from an Electrolyte. This follows at o the principle of the Conservation of Energy if we assume th of Faraday's Law of Electrolysis. Thus suppose for exam t we have a single Daniell's cell placed in series with an electrol l containing acidulated water; then if this arrangement co duce a current which would liberate hydrogen and oxygen fr electrolytic cell for each electro-chemical equivalent of

to $1 \cdot 114 \times 10^8$ mechanical units. Hence we see th
the electrolytic cell were decomposed, $3{\cdot}56 \times 10$
would be gained for each unit of electricity that p
cell: as this is not in accordance with the principle
tion of Energy the decomposition of the water c
see that electrolytic decomposition can only go or
energy in the battery is greater than the gain of ene
lytic cell.

If we attempt to decompose an electrolyte, aci
example, by an insufficient electromotive force the
mena occur. When the battery is first connected to
of electricity runs through the cell, hydrogen tr
current to the plate where the current leaves
travelling up against the current to the other p
hydrogen nor the oxygen, however, is liberated
adheres to the plates, polarizing them and produc
which tends to stop the current; as the current con
amount of gas against the plates increases, and wi
tion, until the E.M.F. of the polarization equals th
when the current sinks to an excessively small frac
value. The current does not stop entirely, a v
continues to flow through the cell. This current
shown by v. Helmholtz to be due to hydrogen and
in the electrolytic cell and does not involve any se
into free hydrogen and oxygen. The way in w
current is carried is somewhat as follows. Suppos
with its small E.M.F. has caused the current to flo
until the polarization of the plates is just sufficie
E.M.F. of the battery; the oxygen dissolved in th
hydrogen coated plate will attack the hydrog

ıus in consequence of the depolarizing action of the dissolv ses there will be a continual current tending to keep the E.M.F. e polarization equal to that of the battery; the current howev not accompanied by the liberation of free hydrogen and oxyg d its production does not violate the principle of the Conservati Energy.

187. Cells in series. When a series of voltaic cel aniell's cells for example, are connected so that the zinc pole e first is joined up to the copper pole of the second, the zinc p the second to the copper pole of the third, and so on, the ce e said to be connected up in series. In this case the total elect otive force of the cells so connected up is equal to the sum of t ectromotive forces of the individual cells. We can see this at or we remember (see Art. 180) that the electromotive force of a stem is equal to the difference between the chemical energy lo hen unit of electricity passes through the system, and the mechani uivalent of the reversible heat generated at junctions of differe bstances: when the cells are connected in series the same chemic anges and reversible heat effects go on in each cell when unit ectricity passes through as when the same quantity of electric sses through the cell by itself, hence the E.M.F. of the cells in ser the sum of the E.M.F.'s of the individual cells.

The resistance of the cells when in series is the sum of th sistances when separate. Thus if E is the E.M.F. and r the resistar a cell, the E.M.F. and resistance of n such cells arranged in ser e respectively nE and nr.

188. Cells in parallel. If we have n similar cells a nnect all the copper terminals together for a new terminal and

consisting of n cells in series, and let these m sets be c
parallel. The E.M.F. of the battery thus formed w
resistance nr/m, where E and r are respectively t
resistance of one of the cells. The current throug
resistance R will be equal to

$$\frac{nE}{R+\frac{nr}{m}}=\frac{E}{\frac{R}{n}+\frac{r}{m}}.$$

Now $nm = N$, hence the denominator of this express
of two terms whose product is given, it will therefore
the terms are equal, i.e. when

$$\frac{R}{n}=\frac{r}{m},$$

or

$$R=\frac{n}{m}r.$$

Since the denominator in this case is as small as possi
will have its maximum value. Since nr/m is the re
battery we see that we must arrange the battery so
possible, the resistance of the battery equal to the
resistance. This arrangement, though it gives the lar
not economical, for as much heat is wasted in the
produced in the external circuit.

190. Distribution of a steady current in Conductors.

Kirchhoff's Laws. The distribution of a stea
a network of linear conductors can be readily determ
of the following laws, which were formulated by Kir

1. The algebraical sum of the currents which me

The second follows at once from the relation (see Art. 180)

$$RI + rI = E,$$

ere R is the external resistance, r the resistance of the batt
ose E.M.F. is E, and I the current through the battery. For
he difference of potential between the terminals of the batte
l by Ohm's law this is equal to the sum of the products of
ength of the current and the resistance for a series of conduct
ming a continuous link between the terminals of the battery.

191. Wheatstone's Bridge. We shall illustrate these la
applying them to the system known as the Wheatstone's Brid
this system a battery is placed in a conductor AB, and five ot
nductors AC, BC, AD, BD, CD are connected up in the w
own in Fig. 91.

Let E be the electromotive force of the battery, B the resista
the battery circuit AB, i.e. the resistance of
battery itself plus the resistance of the wires
nnecting its plates to A and B. Let G be the
istance of CD, and b, a, α, β the resistances of
', BC, AD, BD respectively. Let x be the
rrent through the battery, y the current
ough AC, z that through CD. By Kirch-

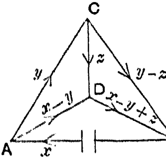


Fig. 91

ff's first law the current through AD will be $x - y$, that thro
$y - z$, and that through DB $x - y + z$.

Since there is no electromotive force in the circuit ACD we h
Kirchhoff's second law,

$$by + Gz - \alpha(x - y) = 0;$$

negative sign is given to the last term because travelling ro

From (1) and (2) we get

$$\frac{x}{G(a+b+\alpha+\beta)+(b+\alpha)(a+\beta)}=\frac{y}{G(\alpha+\beta)+\alpha(a+\beta)}$$

$$=\frac{z}{a\alpha-b\beta} \quad\ldots\ldots\ldots\ldots\ldots(3)$$

Since the electromotive force round the circuit ACB is E,
ave

$$Bx+by+a(y-z)=E;$$

ence by (3), we have

$$\left.\begin{aligned}
x&=\{G(a+b+\alpha+\beta)+(b+\alpha)(a+\beta)\}\frac{E}{\Delta}\\
y&=\{G(\alpha+\beta)+\alpha(a+\beta)\}\frac{E}{\Delta}\\
z&=(a\alpha-b\beta)\frac{E}{\Delta}\\
x-y&=\{G(a+b)+b(a+\beta)\}\frac{E}{\Delta}\\
y-z&=\{G(\alpha+\beta)+\beta(\alpha+b)\}\frac{E}{\Delta}\\
x-y+z&=\{G(a+b)+a(b+\alpha)\}\frac{E}{\Delta}
\end{aligned}\right\}\ldots(4)$$

here

$$= BG(a+b+\alpha+\beta)+B(b+\alpha)(a+\beta)$$
$$+\,G(a+b)(\alpha+\beta)+\alpha(a+\beta)(a+b)-a(a\alpha-$$
$$= BG(a+b+\alpha+\beta)+B(b+\alpha)(a+\beta)$$
$$+\,G(a+b)(\alpha+\beta)+ab\alpha+ab\beta+a\alpha\beta+b\alpha\beta,$$

d two, omitting the product of any two which meet at either e extremities of the battery arm or at either of the extremities e arm P.

From these expressions we see at once that if we keep *all* t istances the same, then the current in one arm (A) due to ctromotive force E in another arm (B), is equal to the current) when the electromotive force E is placed in the arm (A). T ciprocal relation is not confined to the case of six conductors, b true whatever the number of conductors may be.

We may write the expression for x given by equation (4) in t rm

$$x = \frac{E}{B+R},$$

here

$$R = \frac{G(a+b)(\alpha+\beta) + a\alpha\beta + a\alpha b + a\beta b + \alpha\beta b}{G(a+b+\alpha+\beta) + (b+\alpha)(a+\beta)}.$$

R is the resistance, between A and B, of the crossed quad teral $ACBD$.

We see that R = (sum of products of the five resistances of t adrilateral taken three at a time, leaving out the product of a ree that meet in a point): divided by the sum of the products e same resistances taken two at a time, leaving out the prod any pair that meet in A or B.

192. Conjugate Conductors. The current through ill vanish if

$$a\alpha = b\beta;$$

this case AB and CD are said to be conjugate to each other, t e so related that an electromotive force in AB does not prod ny current in CD: it follows from the reciprocal relation that w is is the case an electromotive force in CD will not produce

Since the difference of potential between C and I
that between D and B, we have

$$ay = \beta (x - y);$$

hence eliminating y and $x - y$, we get

$$\frac{b}{a} = \frac{\alpha}{\beta},$$

or

$$b\beta = a\alpha.$$

When this relation holds we may easily prove that

$$\Delta = (a + b + \alpha + \beta) \left\{ G + \frac{(b + \alpha)(a + \beta)}{a + b + \alpha + \beta} \right\}$$

$$\left\{ B + \frac{(\alpha + \beta)(a}{a + b + \alpha} \right.$$

which we may write as

$$\Delta = S (G + P') (B + P);$$

where S is the resistance of ADB, ACB placed in s
resistance of the same conductors when in parallel,
resistance of CAD, CBD in parallel.

When AB is conjugate to CD, then in whatever pa
work an electromotive force is placed, the current th
these arms is independent of the resistance in the oth
deduce this from the preceding expressions for the curre
arms of the circuit; it can also be proved in the followin
is applicable to any number of conductors. Suppose th
motive force in some branch of the system produces a cu
AB, then we may introduce any E.M.F. we please into
altering the current through its conjugate CD. We may
introduce such an electromotive force as would make
through AB vanish, without altering the current in
effect of making the current in AB vanish would be

hus bears to E the ratio of $G(a\alpha - b\beta)$ to Δ. By making $a\alpha -$ all we can without using either very small or very large resistan ke the ratio of the potential difference between C and D to eedingly small; for example, let $a = 101$, $\alpha = 99$, $b = \beta =$ 1 $= G = 1$. Thus we find that this ratio is nearly equal to $1/4 \times$ 1 the potential difference between C and D is only about one fo lionth part of the E.M.F. of the battery.

194. Heat produced in the System of Conducto suming Joule's law (see Art. 178) we shall show that for all poss tributions consistent with Kirchhoff's first law, the one that g minimum rate of heat production is that given by the second l

For, consider any closed circuit in a network of conductors, v, w ... be the currents through the arms of this circuit as de ned by Kirchhoff's laws, and $r_1, r_2, \ldots$ the corresponding resistan e rate of heat production in this closed circuit is by Joule's ual to

$$r_1 u^2 + r_2 v^2 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1$$

Now suppose that the currents in this circuit are altered in st general way possible consistent with leaving the current conductors not in the closed circuit unaltered, and consist o with the condition that the algebraical sum of the curr wing into any point should vanish: we see that these condit quire that all the currents in the closed circuit should be increa diminished by the same amount. Let them all be increased the rate of heat production in the circuit is now by Joule's la

$$(u + \xi)^2 + r_2 (v + \xi)^2 + \ldots$$

$$= r_1 u^2 + r_2 v^2 + \ldots + 2\xi (r_1 u + r_2 v + \ldots) + (r_1 + r_2 + r_3 + \ldots$$

Now since the currents u, v, w are supposed to be determined rchhoff's laws

195. Use of the Dissipation Function. conveniently deduce the actual distribution of the c down F the expression for the rate of heat produ it a minimum, subject to the condition that the the currents which meet in a point is zero. Or w of this condition express, as in the example of Bridge, the current through the various arms in number of currents x, y, z, then express the rate in terms of x, y, z.

F is often called the *Dissipation Function.*

When there are electromotive forces E_p, E_q in which currents u_p, u_q are flowing respectively distribution of current is that which makes

$$F - 2(E_p u_p + E_q u_q + \ldots)$$

a minimum. Thus in the case of the Wheatstone's

$$F = Bx^2 + by^2 + a(y-z)^2 + Gz^2 + \alpha(x-y)^2 -$$

and equations (4) of Art. 191 are equivalent to

$$\frac{d}{dy}(F - 2Ex) = 0,$$

$$\frac{d}{dz}(F - 2Ex) = 0,$$

$$\frac{d}{dx}(F - 2Ex) = 0,$$

which are the conditions that $F - 2Ex$ should be

A very important example of the principle th distribute themselves so as to make the rate of h small as possible, is that of the flow of a stead a uniform wire; in this case the rate of heat produc when the current is uniformly distributed over t

oduction. Let A and B respectively denote the network be d after the increase in resistance in one or more of its arms. W t altering the resistance alter the currents until the distribu currents through A is the same as that actually existing in e rate of heat production for the new distribution is by Art. eater than that in A. Now take this constrained system thout altering the currents suppose that the resistances reased until they are the same as in B. Since the resistances reased without altering the currents the rate of heat product increased, so that as this rate was greater than in A before istances were increased it will *à fortiori* be greater afterwa t after the resistances were increased the currents and resistan the same as B, hence the rate of heat production in B erefore its resistance is greater than that of A.

197. The following proof of the reciprocal relations between rents and the electromotive forces in a network of conductor e to Professor Wilberforce. Let A, B be two of the points i work of conductors, let R_{AB} denote the resistance of the ning AB; V_A the potential of A, V_B that of B, C_{AB} the curr wing along the wire from A to B, E_{AB} the electromotive forc attery in AB, tending to make the current flow through the batt the direction AB; let currents from an external source be led i network, the current entering at a point A being denoted . Then if ΣI_A denotes the sum of all these currents $\Sigma I_A = 0$

We have by Ohm's law,

$$R_{AB}C_{AB} = V_A - V_B + E_{AB} \quad \ldots\ldots\ldots\ldots\ldots(1$$

t us suppose that another distribution of currents, potentials, ctromotive forces is denoted by dashed letters. We have by (

Since the left-hand side is symmetrical
accented and unaccented letters we have

$$\Sigma V_A I'_A + \Sigma E_{AB} C'_{AB} = \Sigma V'_A I_A +$$

Now suppose that all the I's and I''s are
E's are zero except E_{AB}, all the E''s except

$$E_{AB} C'_{AB} = E'_{CD} C_{CD},$$

i.e. that when unit electromotive force acts in
through another branch CD of the network is t
through AB when unit electromotive force
equation (2) suppose that all the E's and E''s a
I_A is led in at A and out at B, all the other
in the distribution represented by the dashed
led in at C and out at D, all the other I''s va

$$I'_C (V_C - V_D) = I_A (V'_A$$

Thus if unit current be led in at A and
difference between C and D is the same as th
between A and B when unit current is led in

198. Distribution of Current throu
ductor. We shall now consider the case
instead of being constrained to flow along wire
themselves through an unlimited conductor
constant throughout its volume. We shall sup
is introduced into this conductor by means o
electrodes, i.e. electrodes made of a material wh
vanishes. The currents will enter and leave t
angles to the electrodes, for a tangential cur
would correspond to a finite tangential electri
ductor and therefore in the electrode, but in th

nditions (1) that it is at right angles to the surfaces A and B, a
that since the current is steady, and there is no accumulation
ctricity at any part of the conductor, the quantity of electric
ich flows into any region equals the quantity which flows o
ence we see that the outward flow over any closed surface enclos
and not B is equal to i, over any closed surface enclosing B a
t A is equal to $-i$, and over any closed surface enclosing neit
both of these surfaces is zero. But the electric intensity, w
e conductor is replaced by air and A has a charge $i/4\pi$ of posit
ctricity, while B has an equal charge of negative electric
isfies exactly the same conditions, which are sufficient to determ
without ambiguity; hence the current in the conductor is eq
the electric intensity in the air and is in the same direction.
e such that the tangent to it at any point is in the direction
e current at that point is called a stream-line. The stream-li
ncide with the lines of force in the electrostatic problem.

199. If q is the intensity of the current at any point P (i.e.
rrent flowing through unit area at right angles to the stream-
P), σ the specific resistance of the conductor, ds an element
e stream-line, then by Ohm's law the E.M.F. between the electro
and B is equal to

$$\int \sigma q ds,$$

e integral being extended from the surface of A to that of B.
s constant, this is equal to

$$\sigma \int q ds.$$

If F is the electric intensity at P in the electrostatic proble
ce $F = q$, the E.M.F. between A and B is equal to

$$\sigma \int F ds;$$

Hence the E.M.F. between A and $B = \frac{\sigma i}{4\pi C}$

or the resistance between A and B is equal to

$$\frac{\sigma}{4\pi C}.$$

We see from this that the resistance of concentric spherical surfaces, whose radii are

$$\frac{\sigma}{4\pi ab}(b-a).$$

The resistance per unit length of a shell o bounded by two coaxial cylindrical surfaces b is equal to

$$\frac{\sigma}{2\pi}\log\frac{b}{a}.$$

The resistance between two spherical elec a and b and whose centres are separated by a very large compared with either a or b, is equa

$$\frac{\sigma}{4\pi}\left\{\frac{1}{a}+\frac{1}{b}-\frac{2}{R}\right\},$$

approximately.

The resistance per unit length between cylindrical wires whose radii are a and b, a a distance R apart, where R is very large co approximately

$$\frac{\sigma}{2\pi}\log\frac{R^2}{ab}.$$

If we have two infinite cylinders, one with E per unit length, the other with the charg B are the centres of the sections of these perpendicular to the axis and P a point in this

at is, they are the series of circles for which A and B are inve nts. The lines of force are the lines which cut these circles nt angles, i.e. they are the series of circles passing through A a

But the lines of force in the electrostatic problem coincide w lines along which the currents flow between two parallel cylind electrodes; hence these currents flow in planes at right angles axes of the cylinders, along the circles passing through the t nts in which these planes intersect the axes of the cylinders.

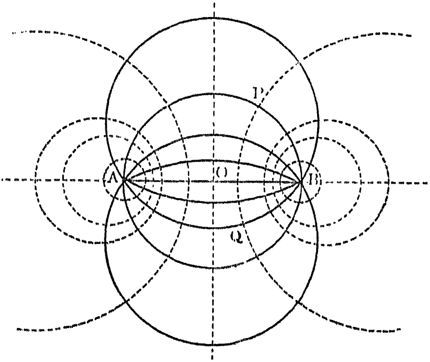


Fig. 92

Since the resistance of unit length of the cylinders is

$$\frac{\sigma}{2\pi}\log\frac{R^2}{ab},$$

resistance of a length t is

$$\frac{\sigma}{2\pi t}\log\frac{R^2}{ab}.$$

is will be the resistance of a thin lamina whose thickness is t w current is led in by circular electrodes radii a and b, if the th

To find the resistance of a lamina so boundec
moment the flow through the unlimited lamina.
flow from out of each electrode approximately
directions; hence if we draw a series of circles in
constant angle α at A and B, we may regard the
made up of the conductors between the stream
multiple arc; the number of these conductors is
same current flows through each, the resistance of
is $2\pi/\alpha$ of the whole resistance; thus the resistanc
conductors is

$$\frac{\sigma}{\alpha t}\log\frac{R^2}{ab}.$$

Thus, for example, if the electrodes are placed
ference of a complete circle, $\alpha = \pi$ and the resistanc

$$\frac{\sigma}{\pi t}\log\frac{R^2}{ab}.$$

200. Conditions satisfied when a curr one medium to another. Let AB be a portio
of separation of two media, σ_1 the specific resista
medium, σ_2 that of the lower, let θ and ϕ be the
directions of the current in the upper and lower m
make with the normal to the surface. Let q_1, q_2
of the currents in the two media, i.e. the amount
across unit areas drawn at right angles to the
Then since, when things are in a steady state, th
or decrease in the electricity at the junction of th
currents along the normal must be equal in the t

Thus $$q_1 \cos\theta = q_2 \cos\phi \quad \ldots\ldots$$

This relation between the directions of the currents in the t
dia is identical in form with that given in Arts. 75 and 158,
relation between the directions of the lines of electric intens
l of magnetic force when these lines pass from one medium
other.

We see that if σ_1 is greater than σ_2, then ϕ is greater than
nce when the current flows from a poor conductor into a be
e the current is bent away from the normal.

The bending of the current as it flows from one medium i
other is illustrated in Fig. 93, which
taken from a paper by Quincke. The
ure represents the current lines in a
cular lamina, one half of which is lead,
other half copper, the electrodes E,
being placed on the circumference.
shows how the currents in going from
worse conductor (the lead) to the
ter one (the copper) get bent away
m the normal to the surface of separa-
n.

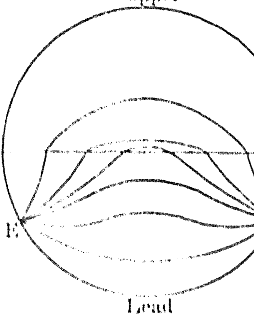


Fig. 93

The electric intensity parallel to the
rmal in the medium whose specific resistance is σ_1 is

$$\sigma_1 q_1 \cos \theta,$$

t in the medium whose specific resistance is σ_2 is $\sigma_2 q_2$ co
ce $q_1 \cos\theta$ is, by equation (1), equal to $q_2 \cos\phi$, we see tha
differs from σ_2 the normal electric intensity will be discontinu
the surface of separation.

If the normal electric intensity is discontinuous there must
istribution of electricity over the surface such that 4π times
face density of this distribution is equal to the discontinuity
normal electric intensity; hence if s is the surface intensity of
ctricity on the surface, and if the current is flowing from the f
dium to the second

CHAPTER X

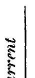
MAGNETIC FORCE DUE TO CURRE

201. It was not known until 1820 that an
exerted any mechanical effect on a magnet in its
year however Oersted, a Professor at Copenhage
magnet was deflected when placed near a wire con
current.

When a long straight wire with a current flowi
held near the magnet, the magnet tended to pla
angles both to the wire and the perpendicular let fa
of the magnet on the wire.

The lines of magnetic force due to a long stra
readily shown by making the wire pass through
board disc over which iron filings are sprinkled. W
right angles to the wire, the iron filings will arra
circles when the current is flowing; these circle
having as their centre the point where the wire cr
the disc.

The connection between the direction of the cu
the magnetic force is such that if the axis of a ri



(i.e. an ordinary corkscrew) c
direction of the current, then
screwed forward into a fixed nu

202. Ampère's law for the magnetic field due to a sed linear circuit. This may be stated as follows: At a nt P, not in the wire conveying the current, the magnetic for e to the current can be derived from a potential Ω where $\Omega = C$ eing the current flowing round the circuit, ω the solid an otended by the circuit at P, and C a constant which depends unit in which the current is expressed.

When the unit of current is what is known as the 'elect gnetic unit,' see Chap. XII, C is unity. We shall in the follow estigations suppose that the current is measured in terms of t t.

We see from Art. 134 that this is equivalent to saying that gnetic field due to a current is the same as that due to a magne ll whose strength is i, the boundary of the shell coinciding w circuit conveying the current. The direction of magnetization shell is related to the direction of the current in such a way t he observer stands on the side of the shell which is charged w sitive magnetism and looks at the current, the current in front flows from right to left.

The best proof of the truth of Ampère's law is that though sequences are being daily compared with the results of exp nts, no discrepancy has ever been detected.

The potential due to the magnetic shell at a point in the substa the shell is not the same as that due to the electric circuit, no magnetic force at such a point the same in the two cases. T wever does not cause any difficulty in determining the magn ce due to a circuit at any point P, for, since only the bound the equivalent magnetic shell is fixed, we can always arrange ll in such a way that it does not pass through P.

shell is magnetized along the normal, the tangent in the shell is equal to the tangential magnetic in Art. 158, the tangential magnetic force at P', a the shell, is equal to the tangential magnetic force inside the shell, and this, as we have just seen tangential magnetic induction at P. Again, by A magnetic force at P' is equal to the normal mag P. Thus since the normal force at P' is equal to th at P, and the tangential force at P' is equal induction at P, the magnetic force at P' is equal direction to the magnetic induction at P. Since at P due to the current is equal to the magnetic the shell, we see that the magnetic force due to t equal to the magnetic induction due to the shell

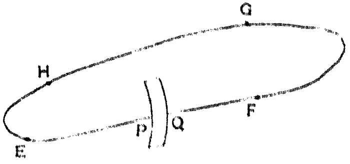


Fig. 95

Thus since the lines of magnetic induction du a series of closed curves passing through the magnetic force due to the current flowing rou circuit will be a series of closed curves threading

203. Work done in taking a magnet closed curve in a magnetic field due to e Let $EFGH$, Fig. 95, be the closed curve traverse

raverses the closed curve $EFGH$ under the influence of
trical currents, is given by the equation

$$W = \int (adx + bdy + cdz),$$

integral being taken round the closed curve.

Hence we have by Art. 153

$$W = \int \{(\alpha + 4\pi A)\, dx + (\beta + 4\pi B)\, dy + (\gamma + 4\pi C)\, dz\},$$

ince by Art. 134 the line integral of the magnetic force du
shell vanishes when taken round a closed circuit, we have

$$\int (\alpha dx + \beta dy + \gamma dz) = 0;$$

ce

$$W = 4\pi \int (Adx + Bdy + Cdz),$$

re the integral is now taken from P to Q, the points where
l cuts the curve $EFGH$, since it is only between P and Q
B, C do not vanish.

If ϕ is the strength of the magnetic shell, and the directio
gration is from the negative to the positive side of the shell

$$\int (Adx + Bdy + Cdz) = \phi;$$

ce

$$W = 4\pi\phi.$$

is the strength of the current which the shell replaces

$$\phi = i,$$

Art. 202; hence

$$W = 4\pi i.$$

Thus the work done on unit pole when it travels round a cl
e which threads the circuit once in the positive direction, i.e. v
pole enters at the negative side of the equivalent shell and le
he positive, is constant whatever be the path, and is equal to

$$4\pi i.$$

negative side and leaves it at the positive), an
negative direction, the work done on the pole on i

$$4\pi i\,(n - m).$$

The value of $\int(\alpha dx + \beta dy + \gamma dz)$ taken round
independent of the nature of the material which i
circuit; it is the same, if the currents are unalt
circuit lies entirely in air, entirely in iron or any
medium, or partly in air and partly in iron. Fo
regarded as made up of two parts, one, in which
the magnetic force are α_1, β_1, γ_1 due to the mag
currents when there is nothing but air in the n
other, a field whose components are α_0, β_0, γ_0 due
tion induced or permanent of the iron.

Hence

$$\int(\alpha dx + \beta dy + \gamma dz) = \int\{(\alpha_1 + \alpha_0)\, dx + (\beta_1 + \beta_0)\,$$

Since α_0, β_0, γ_0 are the forces due to a distr
the work done by these forces on a unit pole tal
circuit must vanish, hence

$$\int(\alpha_0 dx + \beta_0 dy + \gamma_0 dz) = 0,$$

when the integral is taken round any closed circu

$$\int(\alpha dx + \beta dy + \gamma dz) = \int(\alpha_1 dx + \beta_1 dy +$$

and

$$\int(\alpha_1 dx + \beta_1 dy + \gamma_1 dz)$$

$$= 4\pi \text{ (sum of currents embraced by}$$

Thus $\int(\alpha dx + \beta dy + \gamma dz)$ depends merely up
the field and not upon the nature of the material
circuit.

ce the current is infinitely long the magnetic force will not va
we move parallel to the wire conveying the current; hence
rmal magnetic force taken over one of the plane ends will can
t taken over the other. Thus, if S is the curved surface of
inder, the total magnetic force taken over the cylinder is RS, a
ce this vanishes, R must vanish; hence there is no radial magne
ce due to the current.

To find T the tangential magnetic force, let P be any point, a
the perpendicular let fall from P on the current; T is the magne
ce at right angles to OP and to the direction of the current. W
as centre and radius OP describe in a plane at right angles to
rent a circle; at each point on the circumference of this circle
ngential magnetic force will by symmetry be constant, and equa

The work done when unit pole is taken round this circle is 2π
l since the path encircles the current once this must by Art.
equal to $4\pi i$, if i is the strength of the current; hence we have

$$T = \frac{2i}{r},$$

the tangential magnetic force varies inversely as the distance fr
current.

We shall now show that the magnetic force parallel to the curr
nishes.

We can do this by regarding the straight circuit as the limi
ircular one with a very large radius. Consider the magnetic fo
a point P due to the circular current. Through P draw a ci
a plane parallel to that of the current, so that the line join
the centre of this circle, to the centre of the circle in which
rrent is flowing, is perpendicular to the planes of these circ
en if T is the magnetic force along the tangent to this circle

and their planes at right angles to the current. T
magnetic force is related to that of the current in
the diagram, Fig. 94; i.e. the directions of curr
force are related in the same way as the directi
and rotation in a right-handed screw.

The magnetic force at a point P not in the cu
derivable from a potential Ω, where

$$\Omega = 2i\theta + 4\pi n i,$$

where θ is the angle PO, the perpendicular let f
axis of the current, makes with a fixed line in the
right angles to the current: n is an integer. The pot
valued function having at each point an infinit
differing from each other by multiples of $4\pi i$,
done in taking unit magnetic pole round a closed
the current. This indeterminateness in the potent
fact that the work done on unit pole as it goes fr
another point Q, depends not merely on the re
P and Q but also on the number of times the po
P to Q encircles the current.

**205. Magnetic force inside the condu
the current.** When the current is flowi
through a circular cylinder, we can easily find t
at a point inside the cylinder. Let O be the centr
of the conductor, and P a point at which the ta
required; in the plane of the section draw a circ
O and radius OP. The work done in taking un
circle is $2\pi OP \cdot T$, this by Art. 203 is equal to 4π
enclosed by the circle. Hence we have

ll denote by a, the current through the circle whose radius is (
qual to

$$i \cdot \frac{OP^2}{a^2}.$$

Hence

$$2\pi OP \cdot T = 4\pi i \cdot \frac{OP^2}{a^2},$$

$$T = 2i \cdot \frac{OP}{a^2}.$$

Thus when the current is uniformly distributed, the magne
ce inside the cylinder varies directly as the distance from
s; outside the cylinder it varies inversely as this distance.

206. The total normal magnetic induction through any cylind
face passing through two lines parallel to the current is the sa
atever be the shape of the surface
nnecting these lines. This follows at
ce from the principle that the total
gnetic induction over any closed surface
zero. To find an expression for the in-
ction through the cylindric surface, let
and B be the points where the two lines
ersect a plane at right angles to the
rrent, O the point where the axis of the
rrent intersects this plane. Take the

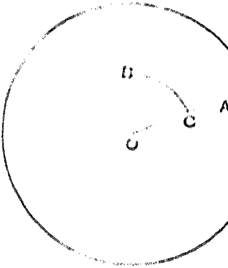


Fig. 96

lindric surface such that if B is the point nearest to O,
rmal section of the surface is the circular arc BC' and
dial portion $C'A$. Since the magnetic force is everywhere tangen
BC' no tube of force passes through the portion corresponding
'; if r is the distance of any point P on $C'A$ from O, the magn
rce at P is

$$\frac{2i}{r};$$

207. Two infinitely long straight flowing in opposite directions. Let A an points where the axes of t a plane drawn at right an of the currents. Let the di at A be downwards thro at B upwards; if i is th

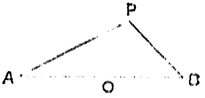


Fig. 97

current, the magnetic potential at a point P is

$$2i\{\angle PAB \pm 2\pi n\} - 2i\{\pi - \angle PBA$$

This may be written

$$4\pi i\,(n + m) - \angle APB \times 2i$$

thus along an equipotential line the angle AP the equipotential lines are the series of circles p

The lines of magnetic force are at right potential lines, they are therefore the series of centres along AB such that the tangents to them point of AB, are of the constant length OA.

The lines of magnetic force and the equ represented in Fig. 98.

The direction of the magnetic force is easi If PT is the direction of the magnetic force at the normal to the circle round APB, the angle complement of the angle PAB.

The magnetic force R at P is the resultant at right angles to AP and $2i/BP$ at right angle these along PT, we have

$$R = \frac{2i}{AP}\cos ABP + \frac{2i}{BP}\cos BA$$

At a point whose distances from A and B are large compa
h AB we may put $AP = BP = OP$, in this case the magne
ce varies inversely as OP^2, and the direction of the force ma
h OP the same angle as OP makes with the line at right ang
AB.

208. Number of tubes of magnetic force due to the t rents which pass through a circuit consisting of t res parallel to the currents. Let A, B be the points where currents intersect a plane drawn at right angles to them, C

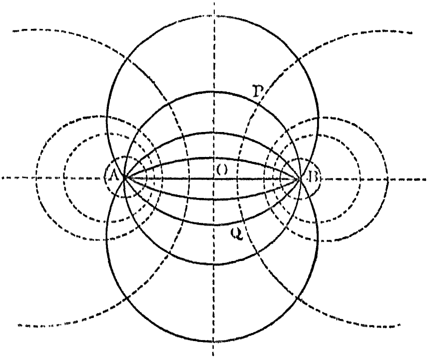


Fig. 98

points where the wires of the circuit cut the same plane. Th t. 206, the number of tubes of magnetic force due to A which p rough CD per unit length $= 2i \log \dfrac{AC}{AD}$. Similarly the num ich pass through CD and are due to the current B is

$$-2i \log \frac{BC}{BD};$$

nce the number through CD per unit length due to the curr

When the circuits AB, CD are so situated th
of tubes passing through CD due to the current
circuits AB, CD are said to be *conjugate* to each c
for this is that $\log \dfrac{AC \,.\, BD}{AD \,.\, BC}$ should vanish, or t

$$\frac{AC}{BC} = \frac{AD}{BD}:$$

another way of stating this result is that C a
points on the same line of magnetic force due
A and B; this is equivalent to the condition th
be points on a line of magnetic force due to
currents at C and D. Since the lines of magne
currents A and B are a series of circles with the
follows that if CD is conjugate to AB it wil
however CD is rotated round the point O', O' be
the line bisecting CD at right angles intersects

A case of considerable practical importance i
equal circuits AB and CD, the current through
direction as that through C and that through B i
as that through D.

Let us consider the case when AB and CD ar
and so placed that the points A, B, D, C are
rectangle. Then if i is the current flowing round
Ω the magnetic potential at a point P will, by A
the equation

$$\Omega = -2i\theta - 2i\phi + \text{constant}$$

where θ and ϕ are the angles subtended respe
CD at P.

The lines of magnetic force are the curves
right angles; along such a line

At a point P on the axis of the current, i.e. on the line thro
at right angles to AB, the magnetic force is parallel to the
d is by Art. 207 equal to

$$\frac{2i\,.\,AB}{AP^2}+\frac{2i\,.\,CD}{CP^2};$$

$OP = x$, $AB = 2a$, $AC = 2d$, the magnetic force at P is equal

$$\frac{4ia}{a^2+(x+d)^2}+\frac{4ia}{a^2+(d-x)^2}.$$

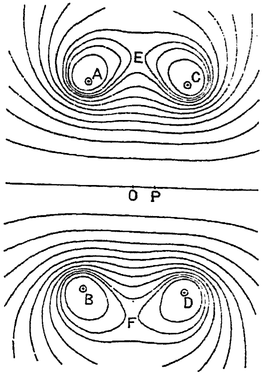


Fig. 99

This is, neglecting the fourth and higher powers of x, equal t

$$\frac{8ia}{a^2+d^2}\left\{1+\frac{3d^2-a^2}{(a^2+d^2)^2}x^2\right\},$$

us, if $\sqrt{3}d = a$, the term in x^2 disappears and the lowest powe
which appears in the expression for the magnetic force is the fou

209. Direct and return currents flow through two parallel and infinite planes.

Let the two parallel planes be at right angles t paper and let this plane the lines AB, CD, Fig. 1 i flow upwards at right an of the paper through eac AB and downwards throug of CD. Let EF be the se parallel to AB and CD and them. We shall prove t force between the planes parallel to EF, being th

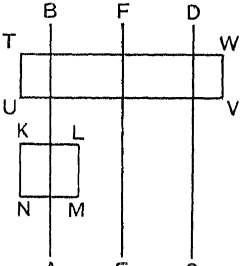


Fig. 100

planes in which the currents are flowing and at r currents.

We shall begin by proving that the magnetic ponent at right angles to the planes in which the cu This is evidently true by symmetry at all points in between AB and CD; we can prove it is true at following way. Take a rectangular parallelepiped is in the plane whose section is EF, let another parallel to the plane of the paper and the third to the line EF. The total normal magnetic in closed surface vanishes. Since the currents are unif in the infinite planes, the magnetic induction will points in a plane parallel to those in which the cur Hence the total magnetic induction over the pai parallelepiped which are at right angles to the pa vanish: for the induction at a point on one face wil

at right angles to the planes in which the currents are flowin proof applies to all parts of the field, whether between t es or outside them.

To prove that the force parallel to the currents vanishes, we ta ctangle $PQRS$ with two sides PQ, RS parallel to the curren other sides PS, QR being at right angles to the planes of t ents. No current flows perpendicularly through this rectang ce (Art. 203) the work done when unit magnetic pole is tak nd its circumference is zero. But since the magnetic force paral PS, RQ vanishes, the work done on unit pole, if F is the for g PQ, F' that along RS, is equal to

$$(F - F')\, PQ.$$

ce this vanishes $F = F'$, i.e. F is constant throughout the fie since by symmetry it vanishes along EF it must vanish throug the field.

We have now proved that throughout the field the componen he magnetic force in two directions at right angles to each otl ish, hence the magnetic force, where it exists, must be paral EF, Fig. 100.

We shall now show that except in the region between the pla force parallel to EF also vanishes. Take a rectangle TUV 100, symmetrical with respect to EF and with the sides T parallel to it. Then since there is as much current flowi vards from AB through this rectangle as downwards from C work done on taking a unit pole round this rectangle m ish. Hence if F is the force parallel to EF along TU, F' that , $F = F'$. If however the directions of all the currents are sed the direction of the magnetic force will be reversed also. F en the currents are reversed the force at TU parallel to EF v

hence $4\pi i \times LM$ is equal to the work done in taki
pole round the rectangle. But this work is $H \times L$
is done when the pole is moving along MN, NK a
have

$$H \times LM = 4\pi i \times LM,$$

or

$$H = 4\pi i.$$

Thus the magnetic force is independent of the
the planes.

210. Solenoid. We can apply exactly th
to the very important case of an infinitely lo
solenoid, i.e. an infinitely long right circular cylin
currents are flowing in planes perpendicular to t
solenoid may be constructed by winding a right
uniformly with wire, the planes of the winding bei
to the axis of the cylinder, so that between any tw
angles to the axis and at unit distance apart there are
of turns of wire. We can show by the same metho
that inside the cylinder the radial magnetic force v
the force parallel to the axis of the cylinder is unif
the cylinder the magnetic force vanishes: and
magnetic force inside the cylinder parallel to the a

$H = 4\pi$ (current flowing between two planes s
distance).

If there are n turns of wire wound round each
cylinder and i is the current flowing through the w
is equivalent to

$$H = 4\pi ni.$$

The preceding result is true whatever be the s

ea about an axis in its own plane which does not cut it, and e ring be wound with wire so that the windings are in pla rough the axis of the ring and so that the number of windi tween two planes which make an angle θ with each other is eq $n\theta/2\pi$; n is thus the whole number of windings on the ring. T can prove as in Art. 209 that the magnetic force vanishes outs e solenoid, and that inside the solenoid the lines of magnetic fo circles having their centres on the axis of the solenoid and th anes at right angles to the axis. Let H be the magnetic force distance r from this axis; the work done on unit pole when tal und a circle whose radius is r and whose centre is on the axis ane perpendicular to it is $2\pi rH$; this by Art. 203 is equal to nes the current flowing through this circle, and is thus equal πni, if i is the current flowing through one of the turns of w nce

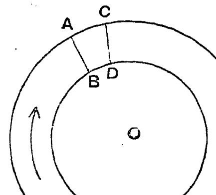
$$2\pi rH = 4\pi ni$$

$$H = \frac{2ni}{r}.$$

Thus the force is inversely proportional to the distance from is.

The preceding proof will apply if the solenoid is wound rou closed iron ring; if however there is a gap in the iron it requ odification.

Let Fig. 101 represent a section of the solenoid and suppose t BDC is a gap in the iron, the faces of the n being planes passing through the axis the solenoid. Let this axis cut the plane the paper in O.



Let P be a point on the face of one of e gaps, B the magnetic induction in the n at right angles to OP, then since the

where θ is the angle subtended by the air gap at
solenoid. Hence by Art. 203 we have

$$rB\left\{\frac{2\pi-\theta}{\mu}+\theta\right\}=4\pi ni$$

or

$$B=\frac{2n\mu i}{r\left\{1+\frac{\theta}{2\pi}(\mu-1)\right\}}.$$

This formula shows the great effect produced
small air gap in diminishing the magnetic inducti
the case of a sample of iron for which $\mu - 1 = 1000$

$$\theta/2\pi = 1/100,$$

i.e. if the air is only one per cent. of the whole circ
B is only one-eleventh of what it would be if the
complete, while even though $\theta/2\pi$ were only equ
magnetic induction would be reduced one-half by
the gap.

We can explain this by the tendency which the t
induction have to leave air and run through iron.
force in the solenoid due to the current circulating
direction of the arrow, the face AB of the gap will
positive magnetism, the face CD with negative. If
of magnetism existed in air, tubes of magnetic in
from AB and running through the air to CD
uniformly distributed in the field; in this case the
in the solenoid for a short part of their course. Bu
solenoid is filled with soft iron these tubes forsake
through the iron, and as they are in the opposite
tubes due to the current they diminish the magnetic

a current i, those on the right relating to the distributio
rrent produced by a battery of electromotive force E.

MAGNETIC SYSTEM.

1. The line integral of the mag-
ic force round any closed curve
eading the magnetizing circuit is
, while round any other closed
ve it vanishes.

2. The lines of magnetic induc-
n are closed curves threading the
gnetizing circuit.

3. The magnetic induction is μ
es the magnetic force, where μ is
magnetic permeability.

4. At the junction of two different
dia the normal magnetic induction
l the tangential magnetic force are
tinuous.

CURRENT SYSTEM.

1. The line integral of the ele
force round any closed curve pas
through the battery is E, while ro
any other closed curve it vanishe

2. The lines of flow of the cur
are closed curves passing through
battery.

3. The intensity of the curre
by Ohm's Law c times the ele
force, where c is the specific con
tivity of the substance, i.e. the
ciprocal of the specific resistance

4. At the junction of two diffe
media the normal electric current
the tangential electric force are
tinuous.

From these results we see that the magnetic induction du
nagnetizing circuit carrying a current i will be numerically e
the current produced by a battery coinciding with the circui
electromotive force of the battery is $4\pi i$, and if the spe
nductivity of the medium at any point in the surrounding f
numerically equal to the magnetic permeability at that point.

Since the magnetic permeability of iron is so much greater t
t of air or other non-magnetic substances, we may, when we
analogy of the current, regard the magnetic substances as g
nductors, the non-magnetic substances as very bad ones.

Thus in the case of a magnetizing coil round an iron ring,
rrent analogue is a battery inserted in a ring of high conductiv

a gap in this circuit, in the electric analogue this
to cutting the ring, inserting a disc of a bad conduct
this would evidently greatly reduce the current; if
the slit, c_1 the specific conductivity of the materia
filled, then the resistance of the ring is $\frac{l-d}{c\pi a^2}+\frac{}{c_1}$
through the ring is equal to

$$\frac{E\pi a^2 c}{l+d\left(\frac{c}{c_1}-1\right)},$$

the average intensity of current is equal to

$$\frac{cE}{l+d\left(\frac{c}{c_1}-1\right)}.$$

The magnetic induction in the slit iron ring will t
magnetic permeability of air is unity, be

$$\frac{4\pi i\mu}{l+d(\mu-1)}.$$

Any problem in the distribution of currents
analogue. Thus take the problem of the Wheats
191), in the magnetic analogue we have six iron b
AD, BD, CD (Fig. 89) with a magnetizing circ
l_1, l_2, l_3, l_4 are the lengths, a_1, a_2, a_3, a_4 the areas of
and $\mu_1, \mu_2, \mu_3, \mu_4$ the magnetic permeability of
respectively, we see from the theory of the Wheat
there will be no lines of magnetic induction down

$$\frac{l_1}{\mu_1 a_1}\ \frac{l_4}{\mu_4 a_4}\qquad \frac{l_2}{\mu_2 a_2}\ \frac{l_3}{\mu_3 a_3},$$

a result which may be applied to the compariso

boundary of the shell exerted a magnetic force at Q equal in θ/PQ^2, where θ is the angle between PQ and the tangent to the boundary of the shell: the direction of the magnetic fo Q is at right angles to both PQ and the tangent to the bound P. Since the magnetic force due to the shell is by Ampère's r same as that due to a current flowing round the boundary of ll, the intensity of the current being equal to the strength of ll, it follows that the magnetic force due to a linear current m calculated by supposing an element of current of length ds to exert at Q a magnetic force equal to $ids \sin\theta/PQ^2$, where i is ength of the current, and θ the angle between PQ and the direct the current at P: the direction of the magnetic force being at ri gles both to PQ and to the direction of the current at P.

The direction of the magnetic force is related to the directio current, like rotation to translation in a right-handed sc rking in a fixed nut.

212. Magnetic force due to a circular current. T eceding rule will enable us to find the magnetic force along is of a circular current.

Let the plane of the current be at right angles to the plane paper. Let the current intersect this plane in the points A, g. 102, flowing upwards at A and downwards B. Let O be the centre of the circle round hich the current is flowing, P a point on the is of the circle. The force at P will by mmetry be along OP. If i is the intensity of current, then the force at P due to an ement ds of the current at A will be at right

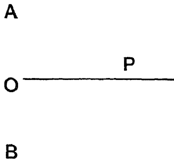


Fig. 102

gles to the current at A, i.e. it will be in the plane of the pa

Thus the force varies inversely as the cube of the
circumference of the circle. At the centre of the
hence the magnetic force at the centre is equal t

$$\frac{2\pi i}{OA},$$

and thus, if the current remains of the same intensit
as the radius of the circle.

The lines of magnetic force round a circular
in Fig. 103. The plane of the current is at right a
of the paper and the current passes through the p

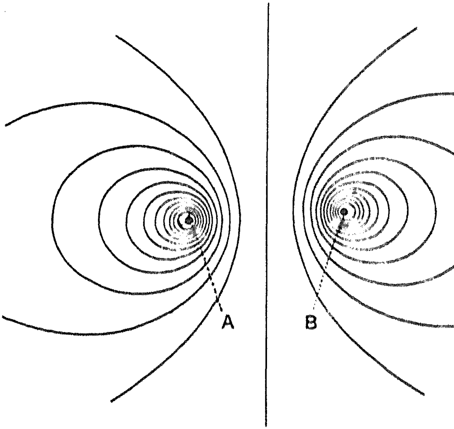


Fig. 103

213. A case of some practical importance is
circular circuits conveying equal currents and plac
coincident. Let A, B; C, D be the points in wh

s midway between the planes of the currents, the magnetic f
P is

$$2\pi i a^2 \left\{\frac{1}{(a^2+(d+x)^2)^{\frac{3}{2}}}+\frac{1}{(a^2+(d-x)^2)^{\frac{3}{2}}}\right\}$$

$$= \frac{4\pi i a^2}{(a^2+d^2)^{\frac{3}{2}}}\left\{1-\tfrac{3}{2}(a^2-4d^2)\frac{x^2}{(a^2+d^2)^2}\right.$$

$$\left.+ \text{ terms in } x^4 \text{ and higher powers of } x\right\}.$$

us if $a = 2d$, that is if the distance between the currents is eq
the radius of either circuit, the lowest power of x in the express
the magnetic force will be the fourth. Thus near O where
all the magnetic force will be exceedingly uniform.

This disposition of the coils is adopted in Helmholtz's Galva
ter.

214. Mechanical Force acting on an electric curr ced in a magnetic field.

The mechanical forces exerted by currents on a magnetic sys
equal and opposite to the forces exerted by the magnetic sys
the currents. Since the forces exerted by the currents on
gnets are the same as those exerted by Ampère's system
gnetic shells, it follows that the mechanical forces on the curr
st be the same as those on the magnetic shells; hence the de
nation of the mechanical forces on a system of currents car
ected by the principles investigated in Art. 136. Introducing
ensity of the current instead of strength of the magnetic shell
from that article that the force in any direction acting o
euit conveying a current i is equal to i times the rate of incre
the number of unit tubes of magnetic induction passing thro
circuit when the circuit is displaced in the direction of the fo

and the magnetic induction is shown in the acc
where the magnetic induction is supposed drawn
plane of the paper.

215. Couple acting on a plane circuit form magnetic field. Let A be the area of intensity of the cu between the norm the circuit and th magnetic inductio unit tubes of magn to the uniform fiel the circuit is iAB the strength of th tion in the unifo does not change

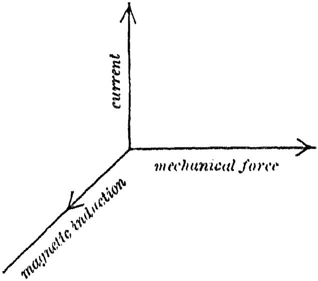


Fig. 104

moved parallel to itself; there are therefore no acting on the system. The number of tubes pa circuit changes however as the circuit is rotate therefore be a couple acting on the circuit; th couple tending to increase ϕ is by the last ar rate of increase with ϕ of the number of unit tube the circuit, that is to

$$\frac{d}{d\phi}(iAB\cos\phi)$$

$$= -iAB\sin\phi.$$

The couple vanishes with ϕ, and hence the cir itself with its normal along the direction of the m and in such a way that the direction of the m

low upwards through the paper. Then, by Art. 204, the magne rce at B due to the current through A is equal to

$$\frac{2i}{AB},$$

d is at right angles to AB; hence, by Art. 214, the mechani rce per unit length on the current at B is equal to

$$\frac{2ii'}{AB},$$

d since it acts at right angles both to the current and to agnetic force, it acts along AB. By the rule given in Art. 2 e see that if the currents are in the same direction the force betw em is an attraction, if the currents are in opposite directions rce between them is a repulsion. Hence, we see that strai arallel currents attract or repel each other, according as they owing in the same or opposite directions, with a force which va versely as the distance between them.

217. Mechanical force between two circuits, ea rcuit consisting of a pair of infinitely long parallel strai onductors. Let the currents be all perpendicular to the pla the paper and let the currents of the first d second pairs intersect the plane of the aper in A, B and C, D respectively: we all consider the case when the circuits are aced symmetrically and so that the line F bisects both AB and CD at right angles. et the current i flow upwards through the

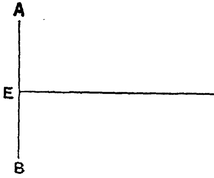


Fig. 105

aper at A, downwards at B, the current i' upwards through aper at C, downwards at D. The force between the circuits

Between the currents B and C there is a repulsio
per unit length to

$$\frac{2ii'}{BC'};$$

the component of this parallel to EF is

$$-\frac{2ii'}{BC'^2}EF.$$

Hence on each unit length of C there is a for
and equal to

$$2ii'EF\left\{\frac{1}{AC'^2}-\frac{1}{BC'^2}\right\};$$

there is an equal force acting in this direction or
of D; hence the total force per unit length on the
attraction parallel to EF equal to

$$4ii'\,EF\left\{\frac{1}{AC'^2}-\frac{1}{BC'^2}\right\}.$$

If $EF=x$, $AE=a$, $CF=b$, this is equal to

$$4ii'x\left\{\frac{1}{(a-b)^2+x^2}-\frac{1}{(a+b)^2+x^2}\right\}$$

this vanishes when $x=0$ and when x is infinite.
be some intermediate value of x when the attractio
This value of x is easily found to be given by the e

$$x^2=\tfrac{1}{3}\{2\sqrt{a^4+b^4-a^2b^2}-(a^2+b^2$$

when $a-b$ is very small this gives

$$x=a-b,$$

when b/a is very small

$$x=\frac{a}{\sqrt{3}}.$$

that between two infinitely long straight parallel circuits, ance between the straight circuits being equal to the short ance between the circular ones. Thus if i, i' are the curre ough the circular circuits, whose radii are respectively a and l x is the distance between the planes of the circuits, the attract ween the parallel circuits is at right angles to the planes of uits and is approximately equal to

$$\frac{4\pi a i i' x}{(a-b)^2 + x^2}.$$

s is a maximum when $x = a - b$; that is, when the dista ween the planes of the circuits is equal to the difference of th ii.

Another case which is easily solved is that of two coaxial circu uits, the radius of one being small compared with that of er. Let i be the intensity of the current flowing round the la uit whose radius is a, i' the current round the small circuit wh ius is b; let x be the distance between the planes of the circu n since b is very small compared with a, the magnetic force the large circuit will be approximately uniform over the sec uit and equal to $2\pi i a^2/(a^2+x^2)^{\frac{3}{2}}$, its value at the centre of t uit. Thus the number of unit tubes of magnetic induction the first circuit which pass through the second circuit is equa

$$\frac{2\pi^2 i a^2 b^2}{(a^2+x^2)^{\frac{3}{2}}}.$$

Hence by Art. 214 the force on the second circuit in the direct which x increases, i.e. the repulsion between the circuits, is equa

$$2\pi^2 i i' a^2 b^2 \frac{d}{dx} \frac{1}{(a^2+x^2)^{\frac{3}{2}}}.$$

a maximum. When we use the attraction betw
means of measuring their intensities, the currents
in this position, for not only is the force to be me
this case, but it is also practically independent of
the proper adjustment of the distance between th

219. Coefficient of Self and Mutual I
coefficient of self-induction of a circuit is defined
of unit tubes of magnetic induction which pass th
when it is traversed by unit current, there being n
permanent magnet in its neighbourhood.

The coefficient of mutual induction of two ci
defined to be the number of unit tubes of magneti
pass through B when unit current flows round
current except that through A, or permanent mag
bourhood of the circuits.

We see from Art. 138 that the coefficient of m
also equal to the number of unit tubes of indu
through A when unit current flows round B.

If the circuit consist of several turns of wire, the
definitions we must take as the number of tubes o
tion which pass through the circuit, the sum of the
of magnetic induction which pass through the diff
circuit.

We see from the preceding definitions that
circuits A and B, and if the currents i, j flow res
these circuits, then the numbers of tubes of magneti
pass through the circuits A and B are respectively

$$Li + Mj, \text{ and } Mi + Nj,$$

d with air, the number of unit tubes of magnetic induc sing through each turn of wire is equal to $4\pi nA$, and since t n turns per unit length, the coefficient of self-induction of a le the solenoid is equal to

$$4\pi n^2 lA.$$

If the core were filled with soft iron of permeability μ, then nber of unit tubes of magnetic induction which pass through n of wire is $4\pi n\mu A$, and the coefficient of self-induction of a le $4\pi n^2 l\mu A$.

If the iron instead of completely filling the core only part it, then if B is the area of the core occupied by the coefficient of self-induction of a length l is

$$4\pi n^2 l\{\mu B + A - B\}.$$

Consider now the coefficient of mutual induction of two solen nd β with parallel axes. The coefficient of mutual induction ish unless one of the solenoids is inside the other, for the magn e due to a current through a solenoid vanishes outside noid. Hence when a current flows through α no lines of induc pass through β unless β is either inside α or completely surro

Let β be inside α. Let B be the area of the solenoid β, an e the number of turns of wire per unit length. Then if rent flows through α, the magnetic force inside is $4\pi n$, where number of turns per unit length. Hence if there is no iron in solenoids, the number of tubes of magnetic induction pas ough each turn of β is $4\pi nB$, and since there are m turns per gth, the coefficient of mutual induction of a length l of the noids is $4\pi nmlB$.

We see, by Art. 218, that the coefficient of mutual induc

and down the other, then by Art. 208, the coe
induction between α and β is, per unit length, e

$$2 \log \frac{AC \,.\, BD}{AD \,.\, BC},$$

where A, B, C, D are respectively the points whe
circuits α and β intersect a plane at right angles
direction. The current through the conductor inte
in A is in the same direction as that through the
through C.

220. We can express the energy in the magn
system of currents very easily in terms of the
coefficients of self and mutual induction of the cir
Art. 163, that the energy per unit length in a unit
at P is equal to $R/8\pi$, where R is the magnetic *for*
of induction is a closed curve, and the total am
this tube is equal to

$$\frac{1}{8\pi} \Sigma R ds,$$

where ds is an element of length of the tube and
sum of all the products Rds for the tube. But
done on unit pole when it is taken round the close
the tube of induction, and this by Art. 203 is equ
sum of the currents encircled by the curve. He
a tube of induction is equal to

$\frac{1}{2}$ (the sum of the currents encircled by t

Hence the whole energy in the magnetic field is
sum of the products obtained by multiplying th
circuit by the number of tubes of magnetic inducti

nce the energy in the magnetic field around this circuit is

$$\tfrac{1}{2}i\,(Li + Mj) + \tfrac{1}{2}j\,(Mi + Nj)$$
$$= \tfrac{1}{2}Li^2 + Mij + \tfrac{1}{2}Nj^2.$$

we have only one circuit carrying a current i, then if L is fficient of self-induction, the energy in the magnetic field is

$$\tfrac{1}{2}Li^2.$$

us the coefficient of self-induction is equal to twice the energy magnetic field due to unit current.

We may use this as the definition of coefficient of self-inducti l this definition has a wider application than the previous c e definition in Art. 219 is only applicable en the currents flow through very fine es, the present one however is applicable en the current is distributed over a con- ctor with a finite cross section. Thus let consider the case where we have a current wing through an infinitely long cylinder ose radius is OA, the direction of flow ng parallel to the axis of the cylinder, l where the return current flows down

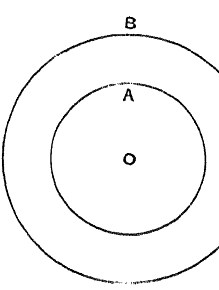


Fig. 106

hin tube, whose radius is OB, coaxial with this cylinder.

Let i be the current which flows up through the cylinder wn through the tube, let us suppose that the current through inder is uniformly distributed over its cross section. The magn ce will vanish outside the tube, for since as much current fl through the cylinder as down through the tube, the total curr wing through any curve enclosing them both vanishes, and th e the work done in taking unit pole round a circle with ce and radius greater than that of the tube will vanish. Since

At a point P inside the cylinder the magnetic f

$$\frac{2ir}{a^2},$$

where $a = OA$, the radius of the cylinder.

By Art. 163 the energy per unit volume is e
where H is the magnetic force; hence if μ is the m
bility of the cylinder, the magnetic energy betwe
right angles to the axis of the cylinder and at unit
equal to

$$\frac{4i^2}{8\pi}\int_{OA}^{OB}\frac{2\pi r dr}{r^2} + \frac{4i^2\mu}{8\pi}\int_0^{OA}\frac{r^2}{a^4}2\pi r dr$$

$$= i^2 \log\frac{OB}{OA} + \frac{i^2}{4}\mu.$$

Hence, since the coefficient of self-induction per un
the energy when the current is unity, it is equal t

$$2\log\frac{OB}{OA} + \tfrac{1}{2}\mu.$$

In this case the coefficient of self-induction wi
greater when the cylinder is made of iron than wl
a non-magnetic metal like copper. For take the case w
where $e = 2{\cdot}718$, the base of the Napierian loga
self-induction for copper, for which μ is equal to
2·5 per unit length, but if the cylinder is made of
whose magnetic permeability is 1000, the coefficient
per unit length is 502. Thus in this case the material
through which the current flows produces an enorn
greater than it does in the case of the solenoids.

The self-induction depends upon the way in w
is distributed in the cylinder; thus if the current ins

221. Rational Current Element. In Ampère's expressi the magnetic force due to a current, the current is supposed divided up into elements, an element ds giving rise to a magne ce equal to $ids \sin \theta/r^2$. Each of those elements when regarded eparate unit corresponds to an unclosed electric current, where the modern theory of electricity such currents do not exi us the mathematical unit does not correspond to a physi lity. To obviate this inconvenience Mr Heaviside has propos other interpretation of the element of current; he points out th e magnetic force $ids \sin \theta/r^2$ is that due to a system of *clo* rrents distributed through space like the lines of magnetic *induct* e to a small magnet, PQ, PQ being the element of current d i representing the number of lines of magnetic induction runni rough PQ, i.e. passing through each cross section of the magn e current at any point in the field round the element of curren presented in magnitude and direction by the *magnetic induct* that point due to the little magnet. The reader will have ficulty in proving this result, if he applies the principle that rk done in taking the unit magnetic pole round any closed circ equal to 4π times the current passing through the circuit. T ement, PQ, *with its associated system of currents*, Mr Heaviside c e *rational current element*, it has the advantage of correspond a possible physical system. It is important to notice that ew of the element of current gives us for closed circuits the sa sult as the old one, i.e. the closed current is entirely confined e closed circuit and does not spread out at all into the surround ace; for let PQ, RS be two elements, then if we place these toget that the end Q of one coincides with the beginning, R, of her, then the analogy with the lines of magnetic induction sh at the currents which when PQ was alone in the field diver

is, that it is a place where n charged particles
direction of the element with the velocity v; n
connected by the relation $nev = ids$.

MEASUREMENT OF CURRENT AND RES

Galvanometers.

222. The magnetic force produced by a curr
measure the intensity of the current. This is m
by means of the tangent galvanometer, which c
coil of wire placed with its plane in the magneti
magnetic field is not wholly due to the earth, t
must contain the resultant magnetic force. At th
there is a magnet which can turn freely about a v
the magnet is in equilibrium its axis will lie al
component of the magnetic force at the centre of
no current is flowing through the coil the axis of
in the plane of the coil. A current flowing th
produce a magnetic force at right angles to the
proportional to the intensity of the current. Let
be equal to Gi where i is the intensity of the curr
the coil and G a quantity depending upon the
coil. G is called the 'Galvanometer constant
horizontal component of the magnetic force at th
Then the resultant magnetic force at the centr
component H in the plane of the coil and a com
angles to it, hence if θ is the angle which the

The smaller we can make H, the external magnetic force at tre of the coil, the larger will be the angle through which a gi rent will deflect the magnet. By placing permanent magnets table positions in the neighbourhood of the coil we can pa tralize the earth's magnetic field at the centre of the coil: in y we can reduce H and increase the sensitiveness of the galva ter. A magnet for this purpose is shown in Fig. 107, which rep ts an ordinary type of galvanometer.

Another method of increasing the sensitiveness of the instrum employed in the 'astatic galvanometer.' In this galvanome g. 108) we have two coils A and B in series, so arranged that rrent circulates round them in opposite directions. Thus, if

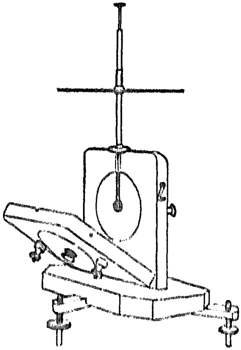


Fig. 107

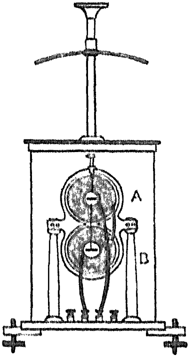


Fig. 108

gnetic force at the centre of the upper coil is upwards from ne of the paper, that at the centre of the lower coil will be do rds. Two magnets α, β, mounted on a common axis, are pla he centres of the coils A and B respectively, the axes of magnet n of these magnets point in opposite directions; thus as

The larger we make G the greater will be th
the galvanometer. If the galvanometer consists o
radius a, then (see Art. 212) $G = 2\pi/a$. If there
together and arranged so that the distance betwe
is a very small fraction of the radius of the turns, t
mately $2\pi n/a$. If the galvanometer consists of
rectangular cross section, the sides of the rectangl
right angles to the plane of the coil, and if $2b$ is th
rectangle (measured at right angles to the plane o
depth in the plane of the coil, n the number of tur
through unit area, then taking as axis of x the
centre of the coil at right angles to its plane, and
through the centre at right angles to this, we hav

$$G = 2\pi n \int_{-b}^{b} \int_{c-a}^{c+a} \frac{y^2 dx dy}{(x^2 + y^2)^{\frac{3}{2}}},$$

where c is the mean radius of the coil.

If 2θ, 2ϕ are the angles subtended at the ce
Fig. 109, this reduces to

$$G = 4\pi n b \log \frac{\cot \frac{\theta}{2}}{\cot \frac{\phi}{2}}.$$

In sensitive galvanometers the hole in the cent
is made as small as possible, so that the inner wi
small radii; when this is the case, we may put ϕ

$$G = 4\pi n b \log \cot \frac{\theta}{2}.$$

rent of this magnitude centuries to liberate 1 c.c. of hydrogen
ctrolysis.

Since

$$i = \frac{H}{G}\tan\theta,$$

$$\frac{\delta\theta}{\delta i} = \frac{G}{H}\cos^2\theta,$$

ile

$$\frac{\delta\theta}{(\delta i/i)} = \sin\theta\cos\theta.$$

Thus for a given absolute increment of i, $\delta\theta$ will be great
en θ is zero, and for a given relative increment, $\delta\theta$, or the cha
deflection, will be greatest when $\theta = 45°$.

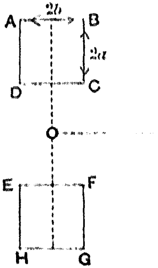


Fig. 109

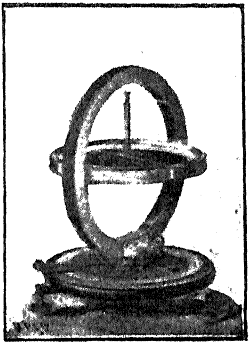


Fig. 110

In some cases it is important to have the magnetic field n
e magnet as uniform as possible. This can be attained (see
3) by using two equal coils placed parallel to one another and
ht angles to the line joining their centres, the distance betw

the coil. When a current passes through the co
deflected out of this plane, and the coil is now m
the axis of the magnet is again in the plane of th
is the case the components of the magnetic force
the plane of the coil due respectively to the cu
external magnetic field must be equal and oppo
external magnetic force, ϕ the angle through which
twisted when the axis of the magnet is again in
coil, the external force at right angles to the pl
$H \sin\phi$. If i is the current through the coil, G the
its centre when the wires of the coil are traversed
then the magnetic force at right angles to the coil d
is Gi; hence when this is in equilibrium with the e
the external field,

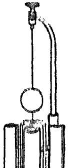
$$H \sin\phi = Gi,$$

or

$$i = \frac{H}{G} \sin\phi.$$

The advantage of this form of galvanometer is
is always in the same position with respect to the c
coils and magnetic field the deflection is greater
for the tangent galvanometer.

224. Desprez-d'Arsonval Galvanomete
vanometer the coil carrying the current moves w



are fixed. The galvanometer is
Fig. 111. A rectangular coil is su
fine metal wires which also serv
current to the coil. The coil mo
poles of a horse-shoe magnet, a
field is concentrated on the coil by

e to the magnetic field. When the magnetic field is uniform ation between the deflection and the current is as follows. be the area of the coil, n the number of turns of wire, i the curr ough the wire, B the magnetic induction at the coil. When ne of the coil makes an angle ϕ with the direction of magne uction the number of tubes of magnetic induction passing throu s

$$BAn \sin \phi,$$

ce, by Art. 215, the couple tending to twist the coil is

$$iBAn \cos \phi.$$

If the torsional couple vanishes when ϕ is zero, the couple w coil is twisted through an angle ϕ will be proportional to ϕ; qual $\tau\phi$, then when there is equilibrium, we have

$$iBAn \cos \phi = \tau\phi,$$

$$i = \frac{\tau\phi}{BAn \cos \phi};$$

ϕ is small this equation becomes approximately

$$i = \frac{\tau\phi}{BAn}.$$

225. Ballistic Galvanometer. A galvanometer may d to measure the total quantity of electricity passing through l, provided the electricity passes so quickly that the magne galvanometer has not time to appreciably change its posi ile the electricity is passing. Let us suppose that when no cur passing the axis of the magnet is in the plane of the coil, the the current passing through the plane of the coil, G the galva eter constant, i.e. the magnetic force at the centre of the coil w

the equation of motion of the magnet during
current is

$$K\frac{d^2\theta}{dt^2} = Gim,$$

thus if the magnet starts from rest the angular ve
t is given by the equation

$$K\frac{d\theta}{dt} = Gm\int_0^t idt.$$

If the total quantity of electricity which passes thr
meter is Q and the angular velocity communicat
ω, we have therefore

$$K\omega = GmQ.$$

This angular velocity makes the magnet swin
of the coil: if H is the external magnetic force at
coil, the equation of motion of the magnet is, if th
force,

$$K\frac{d^2\theta}{dt^2} + mH\sin\theta = 0.$$

Integrating this equation we get

$$K\left\{\left(\frac{d\theta}{dt}\right)^2 - \omega^2\right\} + 2mH(1 - \cos\theta)$$

If ϑ is the angular swing of the magnet, th
vanishes when $\theta = \vartheta$, hence

$$K\omega^2 = 2mH(1 - \cos\vartheta) = 4mH\sin^2$$

On substituting for ω the value previously fou

$$Q = 2\sin\tfrac{1}{2}\vartheta\,\frac{1}{Gm}\sqrt{mH\,.\,K}.$$

If T is the time of a small oscillation of the m

vanometers,' and are constructed so as to make the effects frictional forces as small as possible. This is done either king the moment of inertia of the magnet very large, or king the magnet so symmetrical about its axis of rotation t frictional forces are but small. The correction to be applied w frictional forces are not negligible is investigated in Maxw *ectricity and Magnetism*, Vol. II. p. 386.

226. Measurement of Resistance. The arrangemen nductors in the Wheatstone's Bridge (Art. 191) enables us termine the resistance of one arm of the bridge, say BD, Fig. terms of the resistances of the arms AC, CB and AD. For asurement of resistances by this method wires having a kn istance are used. These are called istance coils, and are made in the lowing way. A piece of silk-covered rman-silver wire is taken and ubled back on itself (to avoid effects to electromagnetic induction, see ap. XI.) and then wound in a coil. length is then carefully adjusted

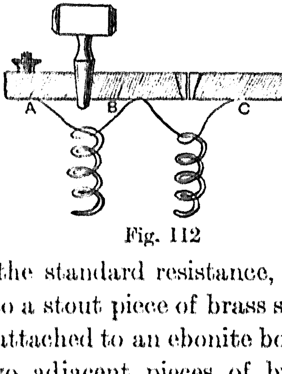


Fig. 112

til its resistance is some multiple of the standard resistance, m. Each end of this coil is soldered to a stout piece of brass s A, B, or C, Fig. 112; these pieces are attached to an ebonite bo insulate them from each other. Two adjacent pieces of b n be put in electrical connection by inserting stout well-fit ass plugs between them. When the plug is out the resista tween B and C is that of the wire, while when the plug is in th practically no resistance between these places.

When there is no current through the arm CD of the Wheatsto

D, and one electrode of a battery to *A*, the other t
ment of the conductors is the same as that in the d
which is reproduced here by the side for conveni
the resistance of *R*: take one or more plugs out o
then proceed to take plugs out of *AD* until there
the galvanometer, when the battery circuit is c
current through *CD* vanishes, we must have by

resistance of *BD* × resistance of *AC*

= resistance of *BC* ×

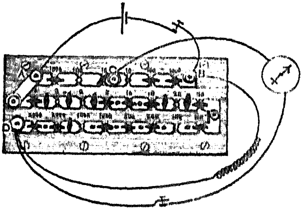


Fig. 113

As the resistances of *AC*, *BC*, *AD* are known, th
mined by this equation.

227. Resistance of a Galvanometer.
Lord Kelvin for measuring the resistance of a g
interesting example of the property of conjugat
saw (Art. 192) that if *CD* is conjugate to *AB*, the
through any arm of the bridge by a battery in *AB*
the resistance in *CD*, and the converse is also true
measure the resistance of a galvanometer, place th
the arm *BD* of the bridge and replace the galva
a key by means of which the circuit *CD* can be co

CHAPTER XI

ELECTROMAGNETIC INDUCTION

228. Electromagnetic Induction, of which the laws were
relled by Faraday, may be illustrated by the following experime
o circuits A and B, Fig. 114, are placed near together, but co
tely insulated from each other; a galvanometer is in the cir
and a battery and key in A. Suppose the circuit A at the beginn
the experiment to be interrupted, press down the key and c

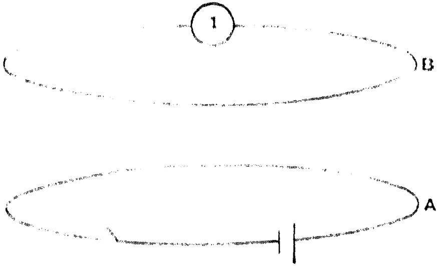


Fig. 114

circuit, the galvanometer in B will be deflected, indicating
ssage of a current through B, although B is completely insula
m the battery. The deflection of the galvanometer is not a p
nent one, but is of the same kind as that of a ballistic galvanome
en a finite quantity of electricity is quickly discharged through
t is, the magnet of the galvanometer is set swinging, but is
rmanently deflected, as it oscillates symmetrically about its

galvanometer in B is again affected, the direction
in this case being opposite to that which occurred
in A was started, indicating that when the curren
an electromotive force is produced round B tending
through B in the same direction as that which pre
A. This electromotive force, like the one produced
A was completed, is but momentary.

These experiments show that the starting or
current in a circuit A is accompanied by the prod
current in a neighbouring circuit B, the current i
opposite direction to that in A when the current
the same direction when the current is stopped.

If instead of making or breaking the current i
is kept steadily flowing in the circuit, while the circu
about, then when A is moving away from B an el
is produced tending to send round B a current in t
as that round A, while if A is moved towards B
force acts round B tending to produce a current
direction to that round A. These electromotive
occur when A is moving, they stop as soon as it is
If we replace the circuit A, with the current flowi
its equivalent magnet, then we shall find that th
magnet will induce the same currents in B as th
circuit A. If we keep the circuit A, or the magnet
B, we also get currents produced in B.

The currents started in B by the alteration
position of the current in A, or by the alteration
B with respect to magnets in its neighbourhood, a
currents; and the phenomenon is called electromag

A good deal of light is thrown on these phenome

the primary one in A, pass through the circuit in the oppo
ection to those sent through it by the current in A; thus
ect of the induced current in B is to tend to make the to
mber of tubes of magnetic induction passing through B zero; t
to keep the total number of tubes of magnetic induction thro
the same as it was before the current was started in A. We sl
d, when we investigate the laws of induction more closely, t
tubes of magnetic induction passing through B, due to
luced current, are at the moment of making the primary circ
ual in number and opposite in direction to those sent thro
by the current in A. The laws of the induction of currents r
us be expressed by saying that the number of tubes of magn
luction passing through B does not change abruptly.

Again, take the case when currents are induced in B by stopp
current in A. Initially the current flowing through A send
mber of tubes of magnetic induction through B: when the curr
A is stopped these tubes cease, but the current induced in B
same direction as that in A causes a number of tubes of magn
luction to pass through B in the same direction as those due
original current in A. Thus the action of the induced curren
ain to tend to keep the number of tubes of magnetic induct
ssing through B constant.

The same tendency to keep the number of tubes of magn
luction through B constant is shown by the induction of a curr
B when A is moved away from or towards B. When A is mo
ay from B, the number of tubes of magnetic induction due
which pass through B is diminished, but there is a current indu
B in the same direction as that through A, which causes additio
bes of magnetic induction to pass through B in the same direct
those due to A; the production of these tubes counterbalan

circuit A itself. Let us suppose that A is alone when a current is flowing round A, tubes of magn through it. If the circuit is broken, and the cu number of tubes would fall to zero; the tend preserve unaltered the number of tubes passing will under suitable circumstances, cause the curr continue flowing in the same direction, to spark when the circuit is broken, even though the orig to send the current through A, was totally ina a spark. To show this effect experimentally it i the coil A round a core of soft iron, so as, with increase the number of tubes of magnetic inducti the circuit; the coil of an electro-magnet shows t

The effect of this very clearly in th ment. The coil of E, Fig. 115, is pla an electric lamp the lamp being ve with that of an consequence of thi connected up to a greater part of th through the coil,

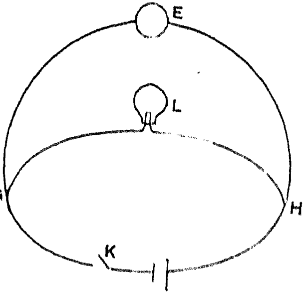


Fig. 115

through the lamp, too little indeed to raise candescence. If however the circuit is broken to keep the number of tubes of magnetic inducti the circuit constant, will send a current mom circuit $HLGE$, which will be larger than that f

tromotive forces due to induction in the two circuits tend
ose each other. When this circuit, connected up to a galva
ter, is placed in a varying magnetic field, no current pass
wing that the electromotive forces in the two circuits are eq
opposite.

Faraday proved that in a magnetic field varying at an assig
e, the electromotive force round a circuit due to induction
portional to the number of tubes of magnetic induction pass
ough the circuit, by taking a coil made of several turns of v
wire, and inserting in it a galvanometer whose resistance
ll compared with that of the coil: when this coil was place
arying field the deflection of the galvanometer was found to
ependent of the number of turns in the coil. As all the resista
the circuit is practically in the coil, the resistance of the circ
be proportional to the number of turns in the coil. Since
antity of electricity passing through the circuit is independen
number of turns, it follows that the E.M.F. round the circ
st have been proportional to the resistance, i.e. to the numbe
ns of the coil. Hence, since the turns of the coils were so c
ether that each enclosed the same number of tubes of magn
uction, it follows that when the rate of change is given the E.
nd the circuit must be proportional to the number of tubes
gnetic induction passing through it.

Faraday also showed by rotating the same circuit at differ
eds in the same magnetic field that the E.M.F. round the cir
proportional to the speed of rotation, i.e. to the rate of cha
the number of tubes of magnetic induction passing through
uit.

These investigations of Faraday's determined the conditi

related to each other like rotation and translatio
screw.

We shall show later on (page 351) that this la
with Ampère's law (Art. 214) by dynamical prin

Let us apply this law of induction to the case
to a variable magnetic field. Let the circuit
battery whose electromotive force is E_0, and l
the circuit, including that of the battery, be R.
of tubes of magnetic induction passing at any
circuit, there will be an E.M.F. equal to $-dP/d$
due to induction; hence by Ohm's law, we hav
round the circuit,

$$Ri = E_0 - \frac{dP}{dt},$$

or

$$\frac{dP}{dt} + Ri = E_0 \quad \text{.........}$$

Suppose the magnetic field is due to two curre
round this circuit and the other through a se
neighbourhood; let j be the current passing
circuit. Let L be the coefficient of self-induction
N that of the second, M the coefficient of mutual
the two circuits. Then as the magnetic field is due

$$P = Li + Mj,$$

and equation (1) becomes

$$\frac{d}{dt}(Li + Mj) + Ri = E_0.$$

If S is the resistance of the second circuit and E_0
force of any battery there may be in that circ
similarly,

$$\frac{d}{}$$

e first and second moving pieces be resisted by resistances p rtional to their velocities, and let $R\dot{x}$, $S\dot{y}$ be these resistan pectively. The momenta corresponding to the two moving pi l be linear functions of the velocities. Let the momentum of st moving piece be

$$L\dot{x} + M\dot{y},$$

at of the second

$$M\dot{x} + N\dot{y}.$$

en, if L, M, N are independent of the coordinates x, y, the eq ns of motion of the two systems will be

$$\frac{d}{dt}(L\dot{x} + M\dot{y}) + R\dot{x} = E_0,$$

$$\frac{d}{dt}(M\dot{x} + N\dot{y}) + S\dot{y} = E_0'.$$

mparing these equations with those for the two currents we at they are identical if we make i, j the currents round the cuits coincide with $\dot{x}$, $\dot{y}$ the velocities of the two moving pie e electrical equations of a system of circuits are thus ident h the dynamical equations of a system of moving bodies, rrent flowing round a circuit corresponds to a velocity, the num tubes of magnetic induction passing through the circuit to omentum corresponding to that velocity, the electrical resista rresponds to a viscous resistance, and the electromotive forc mechanical force.

A further analogy is afforded by the comparison of the Kin ergy of the Mechanical System with the energy in the magn d due to the system of currents. The Kinetic Energy of echanical System is equal to

$$\tfrac{1}{2}\dot{x}(L\dot{x} + M\dot{y}) + \tfrac{1}{2}\dot{y}(M\dot{x} + N\dot{y}).$$

to regard a system of electrical currents as al
The inertia of the system will be increased b
which, for given values of the currents, increases
of electromagnetic induction passing through th
of the system may thus be increased by the int
in the neighbourhood of the circuits.

231. We can illustrate by a mechanical
between the behaviour of electrical circuits and a
system. Models of this kind have been design
Lord Rayleigh; a simple one which serves t
represented in Fig. 116.

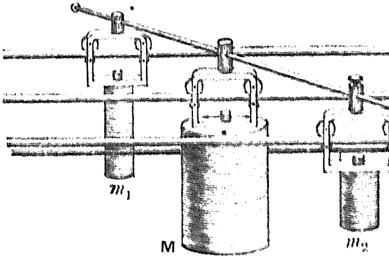


Fig. 116

It consists of three smooth parallel horizonta
masses m_1, M, m_2 slide, the masses being sep
by friction wheels: the three masses are conn
light rigid bar, which passes through holes in sw
upper part of the masses; the bar can slide bac
through these holes, so that the only constraint
is to keep the masses in a straight line.

absence of current in the secondary when the current in mary is uniform. If now we suddenly stop m_1, m_2 will start the direction in which m_1 was moving before being brought t. This is analogous to the direct current in the secondary p ced by the stoppage of the current in the primary. These eff the more marked the greater the mass M.

It is instructive to find the quantities in the dynamical syst ich correspond to the coefficients of self and mutual inducti t us suppose that the bar on which M slides is midway betw other two.

Then if $\dot{x}_1$ is the velocity of m_1 along its bar, $\dot{x}_2$ that of m_2, ocity of M will be $(\dot{x}_1 + \dot{x}_2)/2$, and T the kinetic energy of stem is given by the equation

$$T = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 + \tfrac{1}{2}M\left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2.$$

The momentum along x_1 is $dT/d\dot{x}_1$ and is therefore equal to

$$\left(m_1 + \frac{M}{4}\right)\dot{x}_1 + \frac{M}{4}\dot{x}_2.$$

e momentum along x_2 is $dT/d\dot{x}_2$ and is therefore equal to

$$\frac{M}{4}\dot{x}_1 + \left(m_2 + \frac{M}{4}\right)\dot{x}_2.$$

us $m_1 + M/4$, $m_2 + M/4$ correspond to the coefficients of s luction of the two circuits, while $M/4$ corresponds to the coeffic mutual induction between the circuits. The effect of increa coefficient of mutual induction between the circuits, such crease for example as may be produced by winding the prim d secondary coils round an iron core, may be illustrated by ect produced on the model by increasing the mass M relativel and m_2

to move corresponds to the case when there is n
of the primary circuit a closed circuit round w
circulate: the case when m_2 is fixed correspond
this circuit is broken, when it can produce no el
current can circulate round it. The greater v
m_2 was free than when it was fixed shows that
impulse acts on a circuit the current produced is
is another circuit in the neighbourhood than
circuit was alone in the field; in other words,
secondary diminishes the effective inertia or s
primary.

232. Effect of a Secondary Circuit.

the use of the equations given in Art. 229 we
behaviour of a primary and a secondary coil whe
acts upon the primary. Let us suppose that o
no currents in the circuits. Let L, M, N be respect
of self-induction of the primary, of mutual ind
primary and the secondary, and the coefficient
the secondary: R, S the resistances of the prin
respectively, x and y the currents through these
the external electromotive force acting on the p
the equations of Art. 229,

$$\frac{d}{dt}(Lx + My) + Rx = P' \quad ..$$

$$\frac{d}{dt}(Mx + Ny) + Sy = 0 \ldots..$$

The primary is acted on by an impulse, that i
lasts for a short time, let us call this time τ. T
values of x, y due to this impulse we have by inte
from $t = 0$ to $t = \tau$

$$Lx_0 + My_0 + R\int_0^\tau xdt = \int_0^\tau P'd$$

Since τ is indefinitely small and x is finite

$$\int_0^\tau P' dt = P,$$

n we have $$Lx_0 + My_0 = P \quad \text{.......................(3)}$$

nilarly by integrating (2) we get

$$Mx_0 + Ny_0 = 0 \quad \text{.......................(4)}$$

nce $$x_0 = \frac{P}{L - \frac{M^2}{N}}, \quad y_0 = -\frac{PM}{LN - M^2}.$$

If the secondary circuit had not been present the current in mary due to the same impulse would have been P/L: thus ect of the secondary is to increase the initial current in the prima diminishes its effective self-induction from L to $L - M^2/N$. T an illustration of the effect described in the last article. Equat expresses that the number of tubes of magnetic induction pass rough the second circuit is not altered suddenly by the imp ing on the first circuit.

When the impulse ceases, the circuits are free from exter ces, and the equations for x and y are

$$\frac{d}{dt}(Lx + My) + Rx = 0 \quad \text{....................(5}$$

$$\frac{d}{dt}(Mx + Ny) + Sy = 0 \quad \text{....................(6}$$

Let us now choose as the origin from which time is measu instant when the impulse ceases. Integrate these equations fr 0 to $t = \infty$, then since x and y will vanish when $t = \infty$ we h

$$R\int_0^\infty x dt = Lx_0 + My_0$$

$$= P \text{ by equation (3)},$$

The presence of the secondary increases th
current dies away just after it is started, but di
which the current ultimately dies away.

Integrating (6) from $t = 0$ to $t = \infty$ we find

$$S\int_0^\infty y\,dt = Mx_0 + Ny_0$$

$$= 0 \text{ by equation (4)}$$

hence the total quantity of electricity passing a
the secondary circuit is zero.

To solve equations (5) and (6) put

$$x = A\epsilon^{-\lambda t},$$

$$y = B\epsilon^{-\lambda t};$$

eliminating A and B we find

$$(R - L\lambda)(S - N\lambda) - M^2\lambda^2$$

hence if λ_1, λ_2 are the roots of this quadratic, w

$$x = A_1\epsilon^{-\lambda_1 t} + A_2\epsilon^{-\lambda_2 t},$$

$$y = B_1\epsilon^{-\lambda_1 t} + B_2\epsilon^{-\lambda_2 t}.$$

We notice that since $\frac{1}{2}Lx^2 + Mxy + \frac{1}{2}Ny^2$, th
kinetic energy of the currents, must be positi
x and y, $LN - M^2$ must be positive, and the
positive quantities. If we determine the values
from the values of x and y when $t = 0$, we find a

$$x = \frac{1}{\lambda_1 - \lambda_2}\frac{PN}{LN - M^2}\left\{\epsilon^{-\lambda_1 t}\left(\lambda_1 - \frac{S}{N}\right) - \epsilon^{-\lambda_2 t}\right.$$

$$y = -\frac{1}{\lambda_1 - \lambda_2}\frac{PM}{LN - M^2}\{\lambda_1\epsilon^{-\lambda_1 t} - \lambda_2\epsilon^{-\lambda_2 t}\} \quad ..$$

We see from the quadratic equation (7) tha
greater than, the other root less than S/N, thus

e circuit pass through the other also; this is often expressed ying that there is very little *magnetic leakage* between the circu hen this condition is fulfilled $L - M^2/N$ is very small compa th L. In the limiting case when this quantity vanishes we see uation (7) that one of the values of λ, say λ_2, is infinite, while λ ual to

$$\frac{RS}{LS + NR}.$$

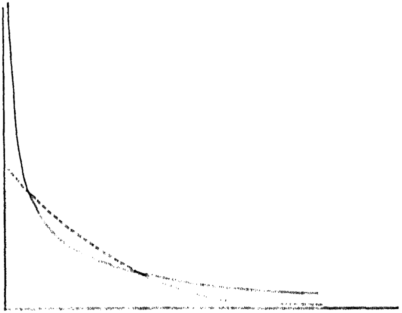


Fig. 117

this case we find from equations (8) and (9) that, except at ry beginning of the motion,

$$x = \frac{P}{L\left\{1 + \frac{RN}{LS}\right\}^2} \epsilon^{-\frac{R}{L}\frac{1}{1+\frac{RN}{LS}}t},$$

$$y = \frac{R}{S}\,\frac{PM}{L^2\left\{1 + \frac{RN}{LS}\right\}^2} \epsilon^{-\frac{R}{L}\frac{1}{1+\frac{RN}{LS}}t}.$$

the case considered in Art. 232. Then using th
article

$$P' = -\frac{dN}{dt},$$

by Faraday's law.

Hence $$P = \int_0^\tau P'dt = -(N_\tau - N_0),$$

where N_τ and N_0 represent respectively the n
magnetic induction passing through the circuit
and $t = 0$ respectively. We have, however, by equ

$$Lx_0 + My_0 \qquad = P,$$

or $$Lx_0 + My_0 + N_\tau = N_0.$$

Now the right-hand side is the number of
induction which pass through the circuit at the
time when the impulse began to act; the left-ha
the number of tubes of magnetic induction, some
due to the currents started in the circuit, which
circuit at the time $t = \tau$ when the impulse ceases t
of these two expressions shows that the current
impulse are such as to keep the number of tubes
tion which pass through the circuit unaltered.
considered is one where there is only one second
is however quite general, and whenever an im
system of conductors, the currents started in th
such that their electromagnetic action causes th
of magnetic induction passing through any of the
to be unaltered by the impulse.

Let us apply this result to the case of the c
a mass of metal by the alteration in an external

en unit pole passes round a closed circuit, is unaltered by t
pulse, the current flowing through any such closed curve is a
altered by the impulse; hence, as there were no currents throu
before the impulse acted, there will be none generated by t
pulse. In other words, the currents generated in a mass of me
an electric impulse are entirely on the surface of the metal, a
inside of the conductor is free from currents.

234. The currents will not remain on the surface, they w
idly diffuse through the metal and die away. We can find
y the currents distribute themselves after the impulse stops
use of the two fundamental principles of electro-dynami
that the work done by the magnetic forces when unit pole trav
nd a closed circuit is equal to 4π times the quantity of curr
wing through the circuit, (2) that the total electromotive fo
nd any closed circuit is equal to the rate of diminution of
nber of tubes of magnetic induction passing through the circ
Let u, v, w be the components of the electric current paralle
axes of x, y, z at any point; α, β, γ the components of the magn
ce at the same point. The axes are chosen so that if x is dra
the east, y to the north, z is upwards. Consider a small rectangu
cuit $ABCD$, the sides AB, BC being parallel to the axes of z
respectively. Let $AB = 2h$, $BC = 2k$. Let α, β, γ be the co
nents of magnetic force at O, the centre of the rectangle; x,
coordinates of O; let the coordinates of P, a point on AB,
$y+k$, $z+\zeta$; the z component of the magnetic force at P wil
proximately

$$\gamma + \frac{d\gamma}{dz}\zeta + \frac{d\gamma}{dy}k.$$

the work done on the pole as it moves from C to

$$-2h\gamma + 2hk\frac{d\gamma}{dy}.$$

We may show similarly that the work done
moves from B to C is equal to

$$-2k\beta - 2hk\frac{d\beta}{dz},$$

and when it moves from D to A, to

$$2k\beta - 2hk\frac{d\beta}{dz}.$$

Adding these expressions we see that the work d
it travels round the rectangle $ABCD$ is equal to

$$\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right)4hk.$$

The quantity of current passing through this rect

$$4uhk,$$

hence since the work done on unit pole in going r
is equal to 4π times the current passing through
Art. 203, we have

$$4\pi \times 4uhk = \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right)4hk,$$

or $$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz} \ldots\ldots\ldots$$

By taking rectangles whose sides are parallel to
z, and of x, y we get in a similar way

$$4\pi v = \frac{d\alpha}{dz} - \frac{d\gamma}{dx} \ldots\ldots\ldots$$

If a, b, c are the components of magnetic induction at O, mber of tubes of magnetic induction passing through the rectan $\times\, 4hk$; hence the rate of diminution of the number of unit tu qual to

$$-\frac{da}{dt}4hk.$$

t by Faraday's law of Electromagnetic Induction the work d unit charge in going round the circuit is equal to the rate ninution in the number of tubes of magnetic induction pass ough the circuit, hence

$$-\frac{da}{dt}4hk = \left(\frac{dZ}{dy} - \frac{dY}{dz}\right)4hk,$$

nilarly

$$\left.\begin{aligned} -\frac{da}{dt} &= \frac{dZ}{dy} - \frac{dY}{dz}, \\ -\frac{db}{dt} &= \frac{dX}{dz} - \frac{dZ}{dx}, \\ -\frac{dc}{dt} &= \frac{dY}{dx} - \frac{dX}{dy} \end{aligned}\right\} \quad \text{..................(4}$$

Let us consider the case when the variable part of magnetiza induced, so that

$$\frac{da}{dt} = \mu\frac{d\alpha}{dt}, \quad \frac{db}{dt} = \mu\frac{d\beta}{dt}, \quad \frac{dc}{dt} = \mu\frac{d\gamma}{dt},$$

here μ is the magnetic permeability. If σ is the specific resist the metal in which the currents are flowing, and if the curr e entirely conduction currents,

$$\sigma u = X, \quad \sigma v = Y, \quad \sigma w = Z.$$

e have by equation (1)

$$4\pi\mu\frac{du}{dt} = \frac{d}{dy}\frac{dc}{dt} - \frac{d}{dz}\frac{db}{dt},$$

hence $$4\pi\mu \frac{du}{dt} = \sigma \left(\frac{d^2u}{dx^2} + \frac{d^2}{dy}\right.$$

similarly $$4\pi\mu \frac{dv}{dt} = \sigma \left(\frac{d^2v}{dx^2} + \frac{d^2}{dy}\right.$$

$$4\pi\mu \frac{dw}{dt} = \sigma \left(\frac{d^2w}{dx^2} + \frac{d}{d}\right.$$

We can also prove by a similar met

$$4\pi\mu \frac{da}{dt} = \sigma \left(\frac{d^2a}{dx^2} + \frac{d^2}{dy}\right.$$

with similar equations for b and c.

These equations are identical in for the conduction of heat, and we see tha force will diffuse inwards into the metal ture would diffuse if the surface of the the heat allowed to diffuse.

235. We may apply the results o heat to the analogous problem in the a simple example let us take a case in o that over the infinite face of a plane sla distribution of currents, and that these c Then from the analogous problem in the that after a time t has elapsed the curr face to which the currents were origina tional to

$$\frac{e^{-\frac{x}{t\sigma/\pi\mu}}}{t^{\frac{1}{2}}}.$$

This expression satisfies the differe

rface, is $2\pi/1600$ seconds, or about 1/250 of a second, a point $\cdot 1$ c
m the surface would receive the maximum current after abo
25,000 of a second, while at a point 10 cm. from the surfa
e current would not reach its maximum for about 4/10 of
ond.

Let us now consider the case of iron: for an average specimen
ft iron we may put $\sigma = 10^4$, $\mu = 10^3$; hence in this case, the ti
e current, 1 cm. from the surface, will take to reach its maximu
lue is about $2\pi/10$ seconds, while a place 10 cm. from the fa
ly attains its maximum after 20π seconds. Thus the curre
fuse much more slowly through iron than they do through copp
e diffusion of the currents is regulated by two circumstances, t
ertia of the currents which tends to confine them to the outs
the conductor, and the resistance of the metal which tends
ake the currents diffuse through the conductor; though the resi
ce of iron is greater than that of copper, this is far more th
unterbalanced by the enormously greater magnetic permeabil
the iron which increases the inertia of the currents, and there
e tendency of the currents to concentrate themselves on the outs
the conductor.

When t is much greater than $x^2/(\sigma/\pi\mu)$, $e^{-\frac{x^2}{t\sigma/\pi\mu}}$ differs little fr
ity, in this case the currents are almost independent of x and va
versely as $t^{\frac{1}{2}}$, thus the currents ultimately get nearly unifor
stributed, and gradually fade away.

**236. Periodic electromotive forces acting on a circ
ossessing inertia.** So far we have confined our attention
e case of impulses; we now proceed to consider the case wh
ectromotive forces act on a circuit for a finite time. If these for
e steady the currents will speedily become steady also, and

vibrations a second, it changes its direction p/π
If i is the current through the coil, we have in the

$$L\frac{di}{dt} + Ri = E.$$

The solution of this equation is, if t is the time
since the application of the electromotive force,

$$i = \frac{E}{R}(1 - e^{-\frac{Rt}{L}}).$$

Thus we see that it is not until t is considerably
that the current approaches the value E/R give
L/R is called the time constant of the coil: for c
metres in length it is only a very small fraction of
large circuits with considerable self-induction it
seconds, while for an iron sphere the size of the
several million years.

When t is small compared with the time consta

$$i = \frac{Et}{L},$$

so that in its initial stages the current does not
resistance but does depend upon the self-inducti
has reached a steady state it does not depend upon
but does depend upon the resistance.

If, after the current has become steady and
electromotive force is removed, we have

$$L\frac{di}{dt} + Ri = 0,$$

or

$$i = \frac{E}{R}e^{-\frac{Rt}{L}},$$

where t is the time which has elapsed since th
electromotive force. Thus it takes a time compara
constant of the coil before the current sinks into

In this case we have,

$$L\frac{di}{dt} + Ri = E\cos pt \qquad \text{.......................(1)};$$

solution of this equation is

$$i = \frac{E\cos(pt - \alpha)}{\{R^2 + L^2p^2\}^{\frac{1}{2}}} \qquad \text{.......................(2)}$$

ere

$$\tan\alpha = \frac{Lp}{R} \qquad \text{.......................(3)}$$

The maximum value of the electromotive force is E, while ximum value of the current is

$$E/\{R^2 + L^2p^2\}^{\frac{1}{2}}:$$

steady force E acted on the circuit the current would be E us the inertia of the circuit makes the maximum current bea maximum electromotive force a smaller ratio than a ste rent through the same circuit bears to the steady electromo ce producing it. The ratio of the maximum electromotive fo the maximum current, when the force is periodic, is equal $+ L^2p^2\}^{\frac{1}{2}}$; this quantity is called the *impedance* of the circui

We see from equation (2) that the phase of the current hind that of the electromotive force. When the force oscillate pidly that Lp is large compared with R, we see from equation at α will be approximately equal to $\pi/2$. In this case the cur rough the coil will be greatest when the electromotive force ac the circuit is zero, and will vanish when the electromotive f greatest.

In this case, since Lp is large compared with R, we have appr ately

$$i = \frac{E}{Lp}\sin pt;$$

this circuit is made of an excellent conductor.
motive forces on the other hand, the current sent
circuit would be enormously greater than that t

The work done by the current per unit tim
heat, is equal to the mean value of either E cos
equal to

$$\frac{1}{2}\frac{E^2R}{R^2+L^2p^2}.$$

Thus when the electromotive force changes so
small compared with R, the work done per unit t
as R; while when the force varies so rapidly th
pared with R, the work done varies directly as I
given the work done is a maximum when

$$R = Lp.$$

237. Circuit rotating in the Earth's f
electromotive force of the type considered in th
duced when a conducting circuit rotates with u
the earth's magnetic field about a vertical axi
the plane of the circuit makes with the magne
horizontal component of the earth's magnetic f
the circuit, then the number of tubes of magnet
through the circuit is

$$HA\sin\theta:$$

the rate of diminution of this is

$$-HA\cos\theta\frac{d\theta}{dt}.$$

If the circuit revolves with uniform angular vel
the rate of diminution in the number of tubes of

e motion of the circuit is resisted by a couple whose moment Art. 214, equal to the current multiplied by the differen efficient with respect to θ of the number of tubes of magn duction due to the earth's field passing through the circuit; t e moment of the couple is

$$iHA\cos\theta,$$

$$\frac{H^2A^2\omega\cos\omega t\cos(\omega t-\alpha)}{\{L^2\omega^2+R^2\}^{\frac{1}{2}}}.$$

Thus the couple always tends to oppose the rotation of the less θ is between $\frac{\pi}{2}$ and $\frac{\pi}{2}+\alpha$ or between $\frac{3\pi}{2}$ and $\frac{3\pi}{2}+\alpha$.

To maintain the motion of the circuit work must be spent; ount of work spent in any time is equal to the mechan uivalent of the heat developed in the circuit.

The mean value of the retarding couple is

$$\frac{1}{2}\frac{H^2A^2\omega\cos\alpha}{\{L^2\omega^2+R^2\}^{\frac{1}{2}}}=\frac{1}{2}\frac{H^2A^2R\omega}{L^2\omega^2+R^2};$$

vanishes when ω is zero or infinite and is greatest when $\omega = I$

If the circuit rotates so rapidly that $L\omega$ is large compared v α is approximately equal to $\pi/2$, and we see that

$$i=-\frac{HA\sin\omega t}{L}.$$

Now by definition Li is the number of tubes of magnetic induct e to the currents which pass through the circuit, while HA sin the number passing through the same circuit due to the ear gnetic field; we see from the preceding expression for i that n of these two quantities, which is the total number of tube gnetic induction passing through the circuit remains zero throu

their coefficient of mutual induction is zero. Th
potential $E \cos pt$ be maintained between the p
see by the preceding investigations that i and j,
two circuits, will be given by the equations

$$i = \frac{E \cos (pt - \alpha)}{\{L^2p^2 + R^2\}^{\frac{1}{2}}},$$

$$j = \frac{E \cos (pt - \beta)}{\{N^2p^2 + S^2\}^{\frac{1}{2}}},$$

where $$\tan \alpha = \frac{Lp}{R}, \quad \tan \beta = \frac{Np}{S}.$$

If the external electromotive force varies so ra
Np are large compared with R and S respectiv

$$i = \frac{E \sin pt}{Lp},$$

$$j = \frac{E \sin pt}{Np},$$

or the currents flowing through the two circuit
portional to their coefficients of self induction
rapidly alternating currents the distribution of th
independent of their resistances and depends alm
self-inductions. Thus if one of the coils had a
the current through the coil would be very
removing the iron, as this would greatly diminis
of the circuit.

239. Transformers. We have hitherto
tion to the case when the only circuit present
upon by the periodic electromotive force. We
the case when in addition to the circuit acted u

romotive force is so large that it would be dangerous to lead t ary circuit about a building; the current for lighting is deriv a secondary circuit consisting of a smaller number of turns

The primary and secondary circuits are wound round an ir as in Fig. 118.

he tubes of magnetic induction concentrate in this core, so th of the tubes which pass through the primary pass also throu econdary.

he current in this secondary is larger than that in the prima the electromotive force acting round it is smaller. The curr e secondary bears to that in the primary approximately

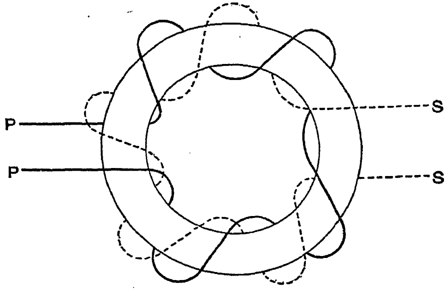


Fig. 118

e ratio as the electromotive force round the primary bear t round the secondary.

Let L, M, N be respectively the coefficients of self-inductio primary, of mutual induction between the primary and ondary and of self-induction of the secondary, let R and resistances of the primary and secondary respectively, x e currents through these coils. Let $E \cos pt$ be the electrom ce acting on the primary. To find x and y we have the follo

By substituting these values in equations (1) an

$$B^2 = \frac{M^2p^2}{N^2p^2 + S^2} A^2 \quad \ldots\ldots$$

$$A^2 = \frac{E^2}{L'^2p^2 + R'^2},$$

$$L' = L - \frac{M^2Np^2}{N^2p^2 + S^2},$$

$$R' = R + \frac{M^2p^2S}{N^2p^2 + S^2},$$

$$\tan \alpha = \frac{L'p}{R'},$$

$$\tan (\beta - \alpha) = \frac{S}{Np}.$$

From the expressions for A and α in terms o
effect of the secondary circuit is to make the pr
like a single circuit whose coefficient of self-indu
resistance is R'. We see from the expressions
L' is less than L, while R' is greater than R.
the secondary circuit diminishes the apparent
primary circuit, while it increases its resistance
motive force changes so rapidly that Np is l
S, we have approximately

$$L' = L - \frac{M^2}{N},$$

$$R' = R + \frac{M^2}{N^2} S,$$

$$B = \frac{M}{N} A,$$

ue of the apparent self-induction of the primary circuit. T
k done per unit time in the transformer is equal to the me
ue of $E \cos pt \,.\, x$, it is thus equal to

$$\frac{1}{2}\frac{E^2 \cos\alpha}{\{L'^2p^2+R'^2\}^{\frac{1}{2}}}$$

$$=\frac{1}{2}\frac{E^2R'}{L'^2p^2+R'^2}.$$

When the secondary circuit is broken S is infinite and theref
$= L$, $R' = R$, and the work done on the transformer per unit ti
the power spent on it is equal to

$$\frac{1}{2}\frac{E^2R}{L^2p^2+R^2}.$$

en the circuit is completed, and S is small compared with N
$L - M^2/N$, $R' = R + M^2S/N^2$, and then the power spent
al to

$$\frac{1}{2}\frac{E^2\left(R+\dfrac{M^2}{N^2}S\right)}{\left(L-\dfrac{M^2}{N}\right)^2p^2+\left(R+\dfrac{M^2}{N^2}S\right)^2}.$$

This is very much greater than the power spent when
ondary circuit is not completed; this must evidently be the ca
when the secondary circuit is completed lamps are raised
andescence, the energy required for this must be supplied to
nsformer. The power spent when the secondary circuit is
npleted is wasted as far as useful effect is concerned, and is sp
heating the transformer. The greater the coefficient of s
luction of the primary, the smaller is the current sent through
mary by a given electromotive force, and the smaller the amo

that is, when

$$S = -\frac{N^2}{M^2}R + \frac{N}{M^2}(LN - M^2)\,p.$$

When there is no magnetic leakage, i.e. when

$$LN = M^2,$$

the power absorbed continually increases as the r
secondary diminishes; when however LN is not e
power absorbed does not necessarily increase as S di
on the contrary reach a maximum value for a par
S, and any diminution of S below this value will l
by a decrease in the energy absorbed by the transform
the frequency of the electromotive force, the lar
resistance of the secondary when the absorption o
transformer is greatest. When the frequency is very
for instance, when a Leyden jar is discharged (see
critical value of the resistance in the secondary may
large. In this case the difference between the maxim
of power and that corresponding to $S = 0$ may be v
when $S = 0$, the power absorbed is equal to

$$\frac{1}{2}\frac{E^2R}{L'^2p^2 + R^2},$$

or approximately for very high frequencies

$$\frac{1}{2}\frac{E^2R}{L'^2p^2},$$

while the maximum power absorbed is

$$\frac{1}{4}\frac{E^2}{L'p},$$

which exceeds that when $S = 0$ in the proportion of

, when Np is large compared with S,

$$\frac{A}{B}=\frac{M}{N},$$

$$\beta-\alpha=\pi.$$

If the primary and secondary coils cover the same length of t
re, and are wound on a core of great permeability, then M/N
qual to m/n, where m is the number of turns in the primary, a
the number in the secondary.

If we have a lamp in the secondary whose resistance is s t
otential difference between its electrodes is sy, i.e.

$$sB\cos(pt-\beta).$$

The maximum value of this expression is sB; substituting t
alue of B, we find that when Np is large compared with S t
alue is equal to

$$\frac{s\dfrac{M}{N}E}{\{L'^2p^2+R'^2\}^{\frac{1}{2}}}.$$

This is greatest when $L'=0$, in which case it is equal to

$$\frac{s\dfrac{M}{N}E}{R'};$$

d this, as S is small compared with Np, is equal to

$$\frac{s\dfrac{M}{N}E}{R+\dfrac{M^2}{N^2}S}.$$

If R is small compared with SM^2/N^2 this is approximately

$$\frac{s}{S}\frac{N}{M}E.$$

electromotive force between the terminals of a
this case always less than 1/20 of the electromotiv
terminals of the secondary. In getting this valu
the conditions to be those most favourable to
a high electromotive force in the secondary; if th
leakage, i.e. if L' is not zero, then at high freq
motive force in the secondary would be very
value just found, in fact where there is any ma
ratio of the electromotive force in the second
primary is indefinitely small when the frequenc

240. Distribution of rapidly alter
When the frequency of the electromotive force
the equations of the type

$$L\frac{dx}{dt} + M\frac{dy}{dt} + \ldots Rx = \text{external electrom}$$

the term Rx depending on the resistance is small
terms Ldx/dt, Mdy/dt depending on induction, w
motive force is supposed to vary as $\cos pt$, wil
Lp, Mp are large compared with R; the equatio
currents take the form

$$\frac{d}{dt}(Lx + My + \ldots) = \text{external electromot}$$

$$= \frac{d\mathbf{N}}{dt},$$

where $\mathbf{N}$ is the number of tubes of induction
system passing through the circuit whose coeffici
is L.

We see from this that

Now consider the case of the currents induced in a mass of met
r a rapidly alternating electromotive force. The number of tub
magnetic induction which pass through any circuit which can
awn in the metal is zero, and hence the magnetic induction mu
nish throughout the mass of the metal. The magnetic force w
nsequently also vanish throughout the same region. But sin
e magnetic force vanishes, the work done when unit pole is tak
und any closed curve in the region must also vanish, and therefo
Art. 203 the current flowing through any closed curve in t
gion must also vanish; this implies that the current vanish
roughout the mass of metal, or in other words, that the curren
nerated by infinitely rapidly alternating forces are confined to t
rface of the metal, and do not penetrate into its interior.

We showed in Art. 235 that the currents generated by
ectrical impulse started from the surface of the conductor a
en gradually diffused inwards. We may approximate to the co
tion of a rapidly alternating force by supposing a series of positi
d negative impulses to follow one another in rapid successio
he currents started by a positive impulse have thus only time
ffuse a very short distance from the surface before the subseque
gative impulse starts opposite currents from the surface; t
fect of these currents at some distance from the surface is to te
counteract the original currents, and thus the intensity of t
rrent falls off rapidly as the distance from the surface of t
nductor increases.

The amount of concentration of the current depends on t
equency of the electromotive force and of the conductivity of t
nductor. If the frequency is infinite and the conductivity fini
r the frequency finite and the conductivity infinite, then the curre
confined to an indefinitely thin skin near the surface of the co

temperature diminish in intensity as we recede
and finally cease to be appreciable. The fluctuati
a long period are appreciable at a greater deptl
a short one. We may for example suppose the t
surface of the earth to be subject to two variations
seasons and having a yearly period, the other dep
of day and having a daily period. These flu
less and less apparent as the depth of the place of
the surface of the earth increases, and finally they
to be measured. The annual variations can, hov
at depths at which the diurnal variations are quit

This concentration of the current near the s
ductor, which is sometimes called 'the throttlin
increases the resistance of the conductor to tl
current. When, for example, a rapidly alternating
along a wire, the current will flow near to the o
and if the frequency is very great the inner part
free from current; thus since the centre of the
current, the current is practically flowing through
a solid wire. The area of the cross section of t
effective in carrying this rapidly alternating curre
than the effective area when the current is cont
case the current distributes itself uniformly over
cross section of the wire. As the effective are
alternating currents is less than that for contin
resistance, measured by the heat produced in u
total current is unity, is greater for the alterna
for continuous currents.

241. Distribution of an alternating cu

face of the conductor. Then if μ is the magnetic permeabil
d σ the specific resistance of the conductor, w the current at
nt x, y, z at the time t parallel to the axis of z, we have by
uations of Art. 234,

$$4\pi\mu\frac{dw}{dt} = \sigma\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\right),$$

since w is independent of y and z,

$$4\pi\mu\frac{dw}{dt} = \sigma\frac{d^2w}{dx^2} \quad\text{......................}(1)$$

We shall suppose that the currents are periodic, making p
mplete alternations per second. We may put, writing i
-1,

$$w = \epsilon^{ipt}\,\omega,$$

ere ω is a function of x, but not of t. Substituting this valu
n equation (1) we get

$$4\pi\mu ip\omega = \sigma\frac{d^2\omega}{dx^2},$$

if $n^2 = 4\pi\mu ip/\sigma$,

$$n^2\omega = \frac{d^2\omega}{dx^2}.$$

e solution of this is

$$\omega = A\epsilon^{-nx} + B\epsilon^{nx},$$

ere A and B are constants.

Now $$n = \left\{\frac{4\pi\mu p}{\sigma}\right\}^{\frac{1}{2}} i^{\frac{1}{2}}$$

$$= \left\{\frac{4\pi\mu p}{\sigma}\right\}^{\frac{1}{2}}\left\{\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right\}$$

words, w cannot be infinite when x is infinite: this co
that B should vanish; in this case we have

$$\omega = A\epsilon^{-\left\{\frac{2\pi\mu p}{\sigma}\right\}^{\frac{1}{2}}x}\epsilon^{-i\left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}}x},$$

and therefore

$$w = A\epsilon^{-\left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}}x}\epsilon^{i\left\{pt-\left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}}x\right\}}.$$

Thus if $w = A \cos pt$ when $x = 0$,

$$w = A\epsilon^{-\left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}}x}\cos\left\{pt - \left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}}x\right.$$

at a distance x from the surface.

This result shows that the maximum value of
a distance x from the face is proportional to $\epsilon^{-\left(\frac{2\pi\mu}{\sigma}\right.}$
magnitude of the current diminishes in geometrica
the distance from the face increases in arithmetical

In the case of a copper conductor exposed to an ele
making 100 alternations per second, $\mu = 1$, $\sigma = 1600$
hence $\{2\pi\mu p/\sigma\}^{\frac{1}{2}} = \pi/2$, so that the maximum current
to $\epsilon^{-\frac{\pi x}{2}}$. Thus at 1 cm. from the surface the ma
would only be ·208 times that at the surface, at a di
only ·043, and at a distance of 4 centimetres less tha
the value at the surface.

If the electromotive force makes a million alterna
$\{2\pi\mu p/\sigma\}^{\frac{1}{2}} = 50\pi$; the maximum current is thus propo
and at the depth of one millimetre is less than one si
of its surface value.

The concentration of the current in the case of ir
remarkable. Consider a sample of iron for which $\mu = 1$

e maximum current at the distance of one-tenth of a millim
m the surface is about one five-hundred-millionth part of
rface value.

We see from the preceding expressions for the current that
tance required to diminish the maximum current to a gi
ction of its surface value is directly proportional to the squ
ot of the specific resistance, and inversely proportional to
uare root of the number of alternations per second.

242. Magnetic Force in the Conductor. The curre
the conductor are all parallel to the axis of z, and are independ
the coordinates y, z.

Now the equations of Art. 234 may be written in the form

$$-\frac{da}{dt}=\sigma\left(\frac{dw}{dy}-\frac{dv}{dz}\right),\quad -\frac{db}{dt}=\sigma\left(\frac{du}{dz}-\frac{dw}{dx}\right),$$

$$-\frac{dc}{dt}=\sigma\left(\frac{dv}{dx}-\frac{du}{dy}\right),$$

ere a, b, c are the components of the magnetic induction, u,
ose of the current. In the case we are considering $u=v=0$,
is independent of y and z; hence $a=c=0$, and the magn
luction is parallel to the axis of y. Thus the currents in the p
accompanied by a magnetic force parallel to the surface of
te and at right angles to the direction of the current.

From the above equations we have

$$\frac{db}{dt}=\sigma\frac{dw}{dx},$$

l by Art. 241 $\qquad w=Ae^{-mx}\cos(pt-mx),$

ere $\qquad m=(2\pi\mu p/\sigma)^{\frac{1}{2}}.$

the conductor carrying the current (see Art. 214). the force is at right angles to the current and als induction, and the magnitude of the force per u conductor is equal to the product of the current induction at right angles to it.

In the case we are considering the magnetic i current are at right angles. If w is the intensity c current flowing through the area $dxdy$ is $wdxdy$ on the volume $dxdydz$ parallel to x, and in the of x, is equal to

$$-wb\,dxdydz.$$

The total force parallel to x acting on the conduct

$$-\iiint wb\,dxdydz,$$

but since b and w are both independent of y and on the conductor per unit area of its face is

$$-\int_0^\infty wb\,dx.$$

Now if α, β, γ are the components of the magn

$$4\pi w = \frac{d\beta}{dx} - \frac{d\alpha}{dy};$$

hence, since $b = \mu\beta$, we see that the force on the to x is

$$-\frac{\mu}{4\pi}\int_0^\infty \beta\frac{d\beta}{dx}\cdot dx$$

$$= \frac{\mu}{8\pi}\{\beta_0{}^2 - \beta_\infty^2\},$$

where β_0 is the value of β when $x = 0$, i.e. at t conductor, and β_∞ is the value of β when $x = \infty$. E

ion of the magnetic field on the currents, be a force due to t
ort of the magnetic substance to move towards the stronger pa
the field. The magnitude of the force parallel to x per unit volu
by Art. 164 equal to $\frac{(\mu-1)}{8\pi}\frac{d\beta^2}{dx}$; thus the force acting per u
a of the face of the slab due to this cause is

$$\int_0^\infty \frac{\mu-1}{8\pi}\frac{d\beta^2}{dx}dx$$

$$= -\frac{(\mu-1)}{8\pi}\beta_0^2.$$

Adding this to the force $\frac{\mu\beta_0^2}{8\pi}$ due to the action of the magn
d on the currents we find that the total force parallel to x is
t area of surface of the slab $\beta_0^2/8\pi$, which for equal values of
the same for magnetic as for non-magnetic substances.

This force is always positive, and hence the conductor tends
ve along the positive direction of x; in other words, the conduc
repelled from the system which induces the currents in the c
ctor. These repulsions have been shown in a very striking w
experiments made by Professor Elihu Thomson and also
Fleming. In these experiments an aluminium plate placed ab
electro-magnet round which a rapidly alternating current
culating, was thrown up into the air, the repulsion between
te and the magnet arising from the cause we have just investiga

The expression $\frac{\beta_0^2}{8\pi}$ is the repulsion at any instant, but since β
portional to $\cos(pt+\epsilon)$ the mean value of β_0^2 is $H^2/2$ if H is
ximum value of β_0. Hence the mean value of the repulsio
ual to

both the current and the magnetic force. Thus action of the currents in the sheet of the condu is d just counterbalances at P the electromag original inducing system situated on the other the conductor.

Hence the slab of thickness d may be regar from P the electromagnetic effect of the origi investigation in Art. 242 we supposed that infinitely thick, but since the currents are pra the slab whose thickness is d, it is evident that t by this layer and that no appreciable advanta creasing the thickness of the slab beyond d. Th slab required to screen off the magnetic force frequency of the alternations and on the magne specific resistance of the conductor. By Arts current and magnetic force at a distance x fr proportional to ϵ^{-mx}, where $m = \{2\pi\mu p/\sigma\}^{\frac{1}{2}}$; h d to reduce the magnetic force to an inapprec surface value md must be considerable. If we r screened off when the magnetic effect is reduced of its undisturbed value, then d the thickness of t proportional to m. The greater the frequency th Thus from the examples given in Art. 241 we se makes a million oscillations a second, a screen a millimetre thick will be perfectly efficient, w a very small fraction of a millimetre in thickness all induction. If the system only makes 100 al the screen if of copper must be several centim several millimetres thick.

tentials of A and B respectively. Then by the laws of the induct currents

$$L\frac{di}{dt} + Ri = \text{electromotive force tending to increase } i$$

$$= V_A - V_B \quad \text{..............................(1).}$$

If Q is the charge on the inside of the jar, and C the capacity e jar, then

$$C(V_A - V_B) = Q,$$

$$(V_A - V_B) = \frac{Q}{C}.$$

The alteration in the charge is due to the current flowing thro e conductor, and i is the rate at which the charge is diminish that

$$i = -\frac{dQ}{dt}.$$

bstituting this value of i in equation (1), we get

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \quad \text{....................(2}$$

he form of the solution of this equation will depend upon whe e roots of the quadratic equation

$$Lx^2 + Rx + \frac{1}{C} = 0$$

e real or imaginary.

Let us first take the case when they are imaginary, i.e. wher

$$R^2 < 4\frac{L}{C}.$$

In this case the solution of (2) takes the form

The charge Q is thus represented by a harmon
amplitude decreases in geometrical progression as
in arithmetical progression.

The discharge of the jar is oscillatory, so that
begin with, the inside of the jar is charged posit
negatively; then on connecting by the circuit t
outside of the jar, the positive charge on the
when however it has all disappeared there is a cur
and the inertia of this current keeps it going,
electricity still continues to flow from the inside o
of positive electricity causes the inside to beco
negative electricity, while the outside gets positive
the jar which had originally positive on the insid
outside, has now negative on the inside, positiv
The potential difference developed in the jar by th
to stop the current and finally succeeds in doin
happens the charges on the inside and outside w
opposite to the original charges if the resistance o
negligible; if the resistance is finite the new c
opposite sign to the old ones, but smaller. The c
to flow in the opposite direction, and goes on flowi
is again charged positively, the outside negatively
resistance the charges on the inside and outside
original values, so that the state of the system wou
when the discharge began; if the resistance is fini
smaller than the original ones. The system goes
until the charges become too small to be apprecia
in the jar and the currents in the wire are thus pe
surging backwards and forwards between the coa

The oscillatory character of the discharge

s air space formed by reflection at a rotating mirror, it will
e discharge is oscillatory, be drawn out into a band with dark a
ght spaces, the interval between two dark spaces depending
e speed of the mirror and the frequency of the electrical vibratio
ddersen observed that the appearance of the image of the
eak formed by a rotating mirror was of this character. He show
oreover that the oscillatory character of the discharge was
oyed by putting a large resistance in the circuit, for he found t
this case the image of the air space was a broad band of lig
adually fading away in intensity instead of a series of bright a
rk bands.

When the discharge is oscillatory the frequency of the dischar
often exceedingly large, a frequency of a million complete osci
ns a second being by no means a high value for such cases.
e by the expression (3) that when $R = 0$, the time of vibratio
$\sqrt{LC}$; thus this time is increased when the self-induction or
pacity is increased. By inserting coils with very great s
luction in the circuit, Sir Oliver Lodge has produced such s
ectrical vibrations that the sounds generated by the success
scharges form a musical note.

In the preceding investigation we have supposed that R^2
s than $4L/C$; if however R is greater than this value, the solut
equation (2) changes its character, and we have now

$$Q = A\epsilon^{-\lambda_1 t} + B\epsilon^{-\lambda_2 t},$$

here $-\lambda_1, -\lambda_2$ are the roots of the quadratic equation

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0.$$

ence $$\lambda_1 = \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{CL}}$$

Hence dQ/dt never vanishes except when $t=0$ and when $t=\infty$. Thus Q which is zero when $t=\infty$ never changes sign. The charge in this case instead of becoming positive and negative never changes sign but continually diminishes, and ultimately becomes too small to be observed. This result is confirmed by Feddersen's observations with the rotating mirror.

The behaviour of the Leyden jar is analogous to that of a mass attached to a spring whose motion is resisted by a force proportional to the velocity. If M is the mass attached to the spring, x the extension of the spring, nx the pull of the spring when the extension is x, rdx/dt the frictional resistance, then the equation of motion of the spring is

$$M\frac{d^2x}{dt^2}+r\frac{dx}{dt}+nx=0.$$

Comparing this with the equation for Q we see that if we compare the extension of the spring to the charge on the jar, then the coefficient of self-induction of the circuit will correspond to the mass attached to the spring, the electrical resistance of the circuit to the frictional resistance of the mechanical system, and the reciprocal of the capacity of the condenser to n, the stiffness of the spring.

The pulling out of the spring corresponds to the charging of the jar, the release of the spring to the completion of the circuit between the inside and the outside of the jar; when the spring is released it will if the friction is small oscillate about its position of equilibrium, the spring being alternately extended and compressed, and the oscillations will gradually die away in consequence of the resistance; this corresponds to the oscillatory discharge of the jar. If however the resistance to the motion of the spring is very great, if for example it is placed in a very viscous liquid like treacle, then when it is released it will move slowly towards its position of equilibrium but will never go through it. This case corresponds to the non oscillatory discharge of the jar when there is great resistance in the circuit.

We have seen that the resistance of a conductor to a variable current is not the same as to a steady one, and thus since the currents which are produced by the discharge of a condenser are not steady, R, which appears in the expression (2), is not the resistance of the circuit to steady currents. Now R the resistance depends upon the frequency of the currents, while as the expression (3) shows, the

requency of the electrical vibrations depends to some extent on the esistance; hence the preceding solution is not quite definite, it epresents however the main features of the case. For a complete solution we may refer the reader to *Recent Researches in Electricity and Magnetism*, J. J. Thomson, Art. 294.

246. Periodic Electromotive Force acting on a circuit containing a condenser. Let an external electromotive force equal to $E \cos pt$ act on the circuit which connects the coatings of the jar, let C be the capacity of the jar, L the coefficient of self-induction, and R the resistance of the circuit connecting its coatings. Then if x is the charge on one of the coatings of the jar (which of the coatings is to be taken is determined by the condition that an increase in x corresponds to a current in the direction of the external electromotive force), we can prove in the same way as we proved equation (2) Art. 245, that

$$L\frac{d^2x}{dt^2} + R\frac{dx}{dt} + \frac{x}{C} = E\cos pt \quad\text{.................(1).}$$

The solution of this equation is

$$x = \frac{E\sin(pt-\alpha)}{p\left\{\left(L-\dfrac{1}{Cp^2}\right)^2 p^2 + R^2\right\}^{\frac{1}{2}}} \quad\text{.................(2),}$$

and thus

$$\frac{dx}{dt} = \frac{E\cos(pt-\alpha)}{\left\{\left(L-\dfrac{1}{Cp^2}\right)^2 p^2 + R^2\right\}^{\frac{1}{2}}} \quad\text{....................(3),}$$

where

$$\tan\alpha = \frac{\left(L-\dfrac{1}{Cp^2}\right)p}{R}.$$

Comparing these equations with those of Art. 234 we see that the circuit behaves as if the jar were done away with and the self-induction charged from L to $L - 1/Cp^2$. We also see from (3) that if Cp^2 is greater than $1/2L$, the current produced by the electromotive force in the circuit broken by the jar (whose resistance is infinite) is actually greater than the current which would flow if the jar were replaced by a conductor of infinite conductivity. If $Cp^2 = 1/L$ the apparent self-induction of the circuit is zero, and the circuit behaves like an inductionless closed circuit of resistance R. Thus by cutting the circuit and connecting the ends to a condenser of

suitable capacity we can increase enormously the current passing through the circuit. We can perhaps see the reason for this more clearly if we consider the behaviour of the mechanical system, which we have used to illustrate the oscillatory discharge of a Leyden jar, viz. the rectilinear motion of a mass attached to a spring and resisted by a frictional force proportional to the velocity. Suppose that X, an external force, acts on this system; then at any instant X must be in equilibrium with (1) the resultant of the rate of diminution of the momentum of the mass, (2) the force due to the compression or extension of the spring, (3) the resistance. If the frequency of X is very great, then for a given momentum (1) will be very large, so that unless (1) is counterbalanced by (2) a finite force of very great frequency will produce an exceedingly small momentum. Suppose however the frequency of the external force is the same as that of the free vibrations of the system when the friction is zero, then when the mass vibrates with this frequency, (1) and (2) will balance each other, so that all the external force has to do is to balance the resistance; the system will therefore behave like one without either mass or stiffness resisted by a frictional force.

247. A circuit containing a condenser is parallel with one possessing self-induction.

Let ABC, AEC, Fig. 119, be two circuits. Let L be the coefficient of self-induction of ABC, R the resistance of this circuit, C the

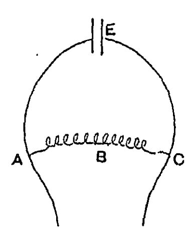


Fig. 119

capacity of the condenser in AEC, r the resistance of wires leading from A and C to the plates. Then if i is the current through ABC, x the charge on the plate nearest to A, we have, neglecting the self-induction of the circuit AEC,

$$L\frac{di}{dt} + Ri = r\frac{dx}{dt} + \frac{x}{C},$$

of these quantities is equal to the electromotive force and C.

$$i = \cos pt,$$

$$x = \frac{(L^2p^2 + R^2)^{\frac{1}{2}}}{\left\{\frac{1}{C^2} + r^2p^2\right\}^{\frac{1}{2}}} \sin(pt + \alpha),$$

$$\alpha = \tan^{-1}\frac{Lp}{R} + \tan^{-1}\frac{1}{rpC}.$$

$$\frac{dx}{dt} = \sqrt{\frac{L^2p^2 + R^2}{\frac{1}{C^2p^2} + r^2}} \cos(pt + \alpha).$$

e maximum current along AEC is to that along ABC as is to $\sqrt{\frac{1}{C^2p^2} + r^2}$, or, if we can neglect the resistances s to the condenser, as $\sqrt{L^2p^2 + R^2} : 1/Cp$. We see that h frequencies practically all the current will go along the ircuit.

ien the frequency is very high a piece of a circuit with rostatic capacity will be as efficacious in robbing neigh-cuits of current as if the places where the electricity s were short-circuited by a conductor.

Lenz's Law. When a circuit is moved in a magnetic h a way that a change takes place in the number of agnetic induction passing through the circuit, a current n the circuit; the circuit conveying this current being in field will be acted upon by a mechanical force. Lenz's that the direction of this mechanical force is such that ends to stop the motion which gave rise to the current. ollows at once from the laws of the induction of currents. o Fig. 120 represents a circuit which, as it moves from , encloses a larger number of tubes of induction passing rom left to right. The current induced will tend to keep of tubes of induction unaltered, so that since the number magnetic induction due to the external magnetic field through the circuit from left to right increases as the es towards the left, the tubes due to the induced current

will pass through the circuit from right to left. Thus the magnetic shell equivalent to the induced current has the positive side on the left, the negative on the right. Since the number of tubes of induction due to the external field which pass through this shell in the negative direction, i.e. which enter at the positive and leave at the negative side, increases as the shell is moved to the left, the force acting on the shell is, by Art. 214, from left to right, which is opposite to the direction of motion of the circuit.

There is a simple relation between the mechanical and electromotive forces acting on the circuit. Let P be the electromotive force, X the mechanical force parallel to the axis of x, i the current flowing round the circuit, u the velocity with which the circuit is moving parallel to x, N the number of unit tubes of magnetic induction passing through the circuit. Then

$$P = -\frac{dN}{dt},$$

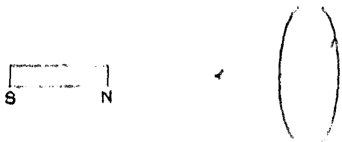


Fig. 120

and if the induced current is due to the motion of the circuit

$$\frac{dN}{dt} = \frac{dN}{dx} \cdot u;$$

hence $$P = -u\frac{dN}{dx}.$$

Again, by Art. 214, we have

$$X = i\frac{dN}{dx},$$

so that $$Xu = -Pi.$$

If we wish merely to find the direction of the current induced in a circuit moving in a magnetic field, Lenz's law is in many cases the most convenient method to use.

An example of this law is afforded by the coil revolving in a magnetic field (Art. 237); the action of the magnetic field on the

urrents induced in the coil produces a couple which tends to stop he rotation of the coil. The magnets of galvanometers are sometimes urrounded by a copper box, the motion of the magnet induces urrents in the copper, and the action of these currents on the nagnets by Lenz's law tends to stop the magnet, and thus brings t to rest more quickly than if the copper box were absent. The uickness with which the oscillations of the moving coil in the Desprez-D'Arsonval Galvanometer (Art. 224) subside is another example of the same effect; when the coil moves in the magnetic ield currents are induced in it, and the action of the magnetic field on these currents stops the coil. Again, if a magnet is suspended over a copper disc, and the disc is rotated, the movement of the disc in the magnetic field induces currents in the disc; the action of the magnet on these currents tends to stop the disc, and there is thus a couple acting on the disc in the direction opposite to its rotation. There must, however, be an equal and opposite couple acting on the magnet, i.e. there must be a couple on the magnet in the direction of rotation of the disc; this couple, if the magnet is free to move, will set it rotating in the direction of rotation of the disc, so that the magnet and the disc will rotate in the same direction. This is a well-known experiment; the disc with the magnet freely suspended above it is known as Arago's disc. Another striking experiment illustrating Lenz's law is to rotate a metal disc between the poles of an electro-magnet, the plane of the disc being at right angles to the lines of magnetic force; it is found that the work required to turn the disc when the magnet is 'on' is much greater than when it is 'off.' The extra work is accounted for by the heat produced by the currents induced in the disc.

249. Methods of determining the coefficients of self and mutual induction of coils. When the coils are circles, or solenoids, the coefficients of induction can be calculated. When, however, the coils are not of these simple shapes the calculation of the coefficients would be difficult or impossible; they may, however, be determined by experiment by means of the following methods.

250. Determination of the coefficient of self-induction of a coil. Place the coil in BD, one of the arms of a Wheatstone's Bridge, and balance the bridge for steady currents, insert

in CD a ballistic galvanometer, and place a key in the battery circuit. When this key is pressed down so as to complete the circuit, although there will be no current through the galvanometer when the currents get steady, yet a transient current will flow through the galvanometer, in consequence of the electromotive forces which exist in BD arising from the self-induction of the coil. This current though only transient is very intense while it lasts and causes a finite quantity of electricity to pass through the galvanometer, producing a finite kick. We can calculate this quantity as follows: an electromotive force E in BD will produce a current through the galvanometer proportional to E, let this current be kE. In consequence of the self-induction of the coil there will be an electromotive force in BD equal to

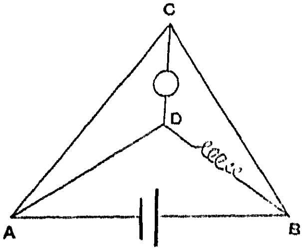


Fig. 121

$$-\frac{d}{dt}(Li),$$

where L is the coefficient of self induction of the coil and i the current passing through the coil. This electromotive force will produce a current q through the galvanometer where q is given by the equation

$$q = -k\frac{d}{dt}(Li).$$

If Q is the total quantity of electricity which passes through the galvanometer

$$\begin{aligned} Q &= \int q\,dt \\ &= -k\int \frac{d}{dt}(Li)\,dt, \end{aligned}$$

the integration extending from before the circuit is completed until after the currents have become steady. The right hand side of this equation is equal to

$$-kLi_0,$$

where i_0 is the value of i when the currents are steady. By the theory of the ballistic galvanometer, given in Art. 225, we see that if θ is the kick of the galvanometer

$$Q = \sin\tfrac{1}{2}\theta \cdot \frac{HT}{\pi G}.$$

here T is the time of swing of the galvanometer needle, G the alvanometer constant, and H the horizontal component of the arth's magnetic force.

Hence we have

$$kLi_0 = \sin \tfrac{1}{2}\theta \cdot \frac{HT}{\pi G} \quad\text{......................(1)}.$$

Let us now destroy the balance of the Wheatstone's Bridge by nserting a small additional resistance r in BD, this will send a urrent p through the galvanometer. To calculate p we notice that he new resistance has approximately the current i_0 running through , and the effect of its introduction is the same as if an electromotive orce ri_0 were introduced into DB, this as we have seen produces current kri_0 through the galvanometer; hence

$$p = kri_0.$$

This current will produce a permanent deflection ϕ of the alvanometer, and by Art. 222

$$p = \tan\phi \frac{H}{G},$$

r

$$kri_0 = \tan\phi \frac{H}{G} \quad\text{........................(2)}.$$

Hence from equations (1) and (2), we get

$$L = r \frac{\sin \frac{1}{2}\theta}{\tan\phi} \frac{T}{\pi}.$$

251. Determination of the coefficient of mutual inuction of a pair of coils. Let A and B, Fig. 122, represent he pair of coils of which A is placed in series with a galvanometer, nd B in series with a battery; this second circuit being provided ith a key for breaking or closing the circuit.

Let R be the resistance of the circuit containing A. Suppose hat originally the circuit containing B is broken and that the key then pressed down, and that after the current becomes steady he current i flows through this circuit. Then before the key is ressed down no tubes of magnetic induction pass through the coil , while when the current i flows through B the number of such nit tubes is Mi, where M is the coefficient of mutual induction etween A and B. Thus the circuit containing A has received an

electrical impulse equal to Mi, so that Q, the quantity of electricity flowing through the galvanometer, will be Mi/R, and if θ is the kick of the galvanometer, we have

$$\frac{Mi}{R} = \sin \tfrac{1}{2}\theta \frac{HT}{\pi G} \quad \text{..........................(1)},$$

using the same notation as before. We can eliminate a good many of the quantities by a method somewhat similar to that used in the last case. Cut the circuit containing the coil A and connect its ends to two points on the circuit B separated by a small resistance S; then if R is very large compared with S this will not alter appreciably the current flowing round B; on this supposition the current flowing round the galvanometer circuit will be

$$\frac{S}{R+S} i,$$

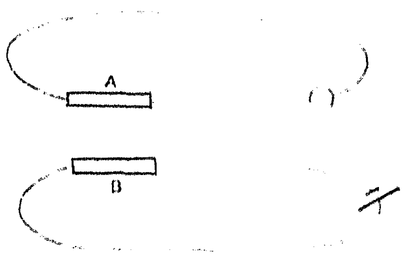


Fig. 122

and if ϕ is the corresponding deflection of the galvanometer

$$\frac{S}{R+S} i = \tan \phi \, . \frac{H}{G} \quad \text{..........................(2)}.$$

Hence from equations (1) and (2), we get

$$M = \frac{RS}{R+S} \frac{\sin \frac{1}{2}\theta}{\tan \phi} \frac{T}{\pi}.$$

252. Comparison of the coefficients of mutual induction of two pairs of coils. Let A, a be one pair of coils, B, b the other. Connect a and b in one circuit with the battery, and connect the points P and Q (Fig. 123) to the two electrodes of a ballistic galvanometer. Insert resistances in PAQ and PBQ until there is no kick of the galvanometer when the circuit through a and b is made

broken. Let R be the resistance then in PAQ, S that in PBQ, d let M_1, M_2 be the coefficients of mutual induction between the ils Aa, Bb respectively, then

$$\frac{M_1}{R} = \frac{M_2}{S}.$$

To prove this we notice that, by Art. 190, if we have any closed rcuit consisting of various parts, the sum of the products obtained multiplying the resistance of each part by the current passing rough it is equal to the electromotive force acting round the rcuit. In the case when the electromotive forces are transient, we t by integrating this result, that the sum of the products got by ultiplying the resistance of each part of the circuit by the quantity electricity which has passed through it is equal to the electro-

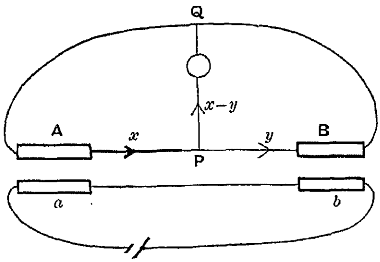


Fig. 123

motive impulse acting round the circuit. Let us apply this to our ase: if i is the steady current flowing through the coils a and b, he electromotive impulse acting on A due to the closing of the ircuit is $M_1 i$, while that on B is $M_2 i$. If x is the quantity of lectricity which passes through A when the circuit through a, b is losed, y that through B, $x - y$ will be the quantity which passes hrough the galvanometer; hence applying the above rule to the ircuit APQ, we have if K is the resistance of the galvanometer circuit

$$Rx + K(x - y) = M_1 i.$$

Applying the same rule to the circuit BPQ, we get

$$Sy + K(y - x) = M_2 i.$$

But if the total quantity which passes through the galvanometer is zero, we have $x = y$, and therefore

$$\frac{M_1}{R} = \frac{M_2}{S}.$$

253. Comparison of the coefficients of self-induction of two coils. Place the two coils whose coefficients of self-induction are L and N respectively in the arms AD, BD of a Wheatstone's Bridge, Fig. 121, balanced for steady currents, then adjust the resistances in AD, BD so that no kick of the galvanometer occurs when the battery circuit is made; these alterations in the resistances of AD and BD will entail proportional alterations in those of AC and BC in order to keep the bridge balanced for steady currents. Then when there is no kick of the galvanometer when the circuit is made, and no steady deflection when it is kept flowing, we have

$$\frac{L}{N}=\frac{P}{Q}=\frac{R}{S},$$

where P, Q, R, S are the resistances of the arms AD, BD, AC, BC respectively.

We can see this as follows: suppose we have a balanced Wheatstone's Bridge with these resistances, then for steady currents the balance will be undisturbed if P and Q are altered in such a way that their ratio remains unchanged; but the alteration of P and Q in this way is equivalent to the introduction into AD and BD of electromotive forces proportional to P and Q. For since no current flows through the galvanometer the same current flows through AD as through BD, and the preceding statement follows by Ohm's Law. Hence we see that the introduction into the arms AD and BD of electromotive forces proportional to P and Q, will not alter the balance of the bridge, and, conversely, that if this balance is not altered by the introduction of an electromotive force A into the arm AD, and another, B, into the arm BD, then A/B must be equal to P/Q.

Now if we have coils in AD and BD whose coefficients of self-induction are L, N, then since after the current gets steady, the same current, i say, flows through each of these coils, there must be, whilst the current is getting steady, an impulse Li in AD, and another equal to Ni in BD. Since these impulses do not send any electricity through the galvanometer they must, by the preceding reasoning, be proportional to P and Q, hence

$$\frac{L}{N}=\frac{P}{Q}.$$

254. Heat developed by the hysteresis of iron. We an, as Dr John Hopkinson showed, deduce from the law of Electro-agnetic Induction the expression given on p. 194 for the heat roduced in iron per unit volume when the magnetic force undergoes cyclical change. Take the case of a solenoid filled with iron and arrying a current whose value i is changing cyclically; let l be the ngth of the solenoid, n the number of turns of wire per unit length, the area of cross section of the core and B the magnetic induction. he electromotive force in the solenoid due to induction is $-nla\frac{dB}{dt}$, ence the work spent by the current in time T in consequence of ie presence of the iron is

$$\int_0^T inla\frac{dB}{dt}\,.\,dt.$$

But if H is the magnetic force

$$H = 4\pi ni,$$

that the work spent by the current, appearing as heat in the iron, equal to

$$\frac{1}{4\pi}\int_0^T laH\frac{dB}{dt}\,dt.$$

Since the volume of the iron is la, the heat produced per unit olume is

$$\frac{1}{4\pi}\int H\frac{dB}{dt}\,dt$$

$$= \frac{1}{4\pi}\int HdB.$$

his is the value already obtained on p. 194.

CHAPTER XII

ELECTRICAL UNITS: DIMENSIONS OF ELECTRICAL QUANTITIES

255. In Art. 9 we defined the unit charge of electricity, as the charge which repelled an equal charge with unit mechanical force when the two charges were at unit distance apart and surrounded by air at standard temperature and pressure. When we know the unit charge the various other electrical units easily follow. Thus the unit current is the one that conveys unit charge in unit time; unit electric intensity is that which acts on unit charge with unit mechanical force; unit difference of potential is the potential between two points when unit work is done by the passage of unit charge from one point to the other. Unit resistance is the resistance between two points of a conductor between which the potential difference is unity when the conductor is traversed by unit current.

The step from the electrical to the magnetic quantities is made by means of the law that the work done when unit magnetic pole is taken round a closed circuit is equal to 4π times the current flowing through the circuit. This law is to some extent a matter of definition. All that is shown by experiment is that the work done when unit pole is taken round the circuit is proportional to the current flowing through the circuit, and, as long as the current remains the same, is independent of the nature of the substances passed through by the pole in its tour round the circuit. If we said that p times the work done was equal to 4π times the current, these conditions would still be fulfilled provided p was independent of the current, the magnetic force and the nature of the substances in the field. Though, as we shall see later, it would be possible to get a somewhat more symmetrical system of units by a proper choice of p, yet in practice, to avoid the introduction of an unnecessary constant, p is always taken as unity. When $p = 1$, it follows from Art. 210 that the magnetic force at the centre of a circle of radius a traversed by a current i is $2\pi i/a$; thus unit magnetic force will be the force at the centre of a circle of radius 2π traversed by unit current.

Thus knowing the unit current we can at once determine the unit magnetic force. Having got the unit magnetic force, the unit magnetic pole follows at once, since it is the pole which is acted on by unit magnetic force with the unit mechanical force. From these units we can go on and deduce without ambiguity the units of the other magnetic quantities. The System of units arrived at in this way is called the Electrostatic System of Units.

Starting from the unit charge as defined in Art. 9, we thus arrive at a unit magnetic pole. In Art. 114, however, we gave another definition of unit magnetic pole deduced from the repulsion between two similar poles. The unit magnetic pole as defined in Art. 114 does not coincide with the unit pole at which we arrive, starting, as we have just done, from the unit charge of electricity. The numerical relation between the two units depends upon what units of length and time we employ; if these are the centimetre and second, then the unit magnetic pole on the electrostatic system of units is about 3×10^{10} times as great as the unit pole defined in Art. 114.

Instead of starting with unit charge of electricity we may start with unit magnetic pole as defined in Art. 114. The units of the other magnetic quantities would at once follow from considerations similar to those by which we deduced the unit electrical quantities from the unit electrical charge. The electrical units would follow from the magnetic ones, by the principle that the magnetic force at the centre of a circular current of radius a is $2\pi i/a$, where i is the strength of the current; thus the unit current is that which produces unit magnetic force at the centre of a circle whose radius is 2π. In this way we can get the unit current, and from this the units of the other electrical quantities follow without difficulty. The System of units got in this way is called the Electromagnetic System of Units.

The electromagnetic system of units does not coincide with the electrostatic system. The electromagnetic unit charge of electricity bears to the electrostatic unit charge a ratio which depends on the units of length and time; if these are the centimetre and second the electromagnetic unit of electricity is found to be about 3×10^{10} times the electrostatic unit. The ratio of the electromagnetic unit of charge to the electrostatic unit is equal to the ratio of the electrostatic unit pole to the electromagnetic unit.

In the following table the relations between the electrostatic and electromagnetic units of various electric and magnetic quantities are given. Here v is the ratio of the electromagnetic unit charge of electricity to the electrostatic unit.

Quantity	Symbol	Electrostatic unit in terms of Electromagnetic
Quantity of Electricity ...	e	$1/v$
Electric intensity	F	v
Potential difference ...	V	v
Current	i	$1/v$
Resistance of a conductor	R	v^2
Electric Polarization ...	D	$1/v$
Capacity of a condenser ...	C	$1/v^2$
Strength of Magnetic Pole	m	v
Magnetic force	H	$1/v$
Magnetic induction ...	B	v
Magnetic permeability ...	μ	v^2
Coefficient of Self-Induction	L	v^2

Certain combinations of these quantities are equal to purely geometrical or dynamical quantities, such as length, force, energy. The numerical expression of such combinations must evidently be the same whatever system of units we employ; thus, for example, the mechanical force on a charge e placed in a field of electric intensity is Fe, but this force is a definite number of dynes, quite independent of any arbitrary system of measuring electric quantities, thus $F \times e$ must be the same whatever system of electrical units we employ.

The following are examples of such combinations.

$$\text{Time} = \frac{e}{i}.$$

$$\text{Length} = \frac{V}{F}.$$

$$\text{Force} = Fe;\ mH.$$

$$\text{Energy} = \tfrac{1}{2}Ve;\ \tfrac{1}{2}\frac{e^2}{C};\ Ri^2t;\ \tfrac{1}{2}Li^2.$$

$$\text{Energy per unit volume} = FD/8\pi;\ \mu H^2/8\pi.$$

Thus since Fe is independent of the electrical units chosen, if we adopt a new system in which the unit of e is v times the old unit, the new unit of F must be $1/v$ times the old unit. Again, Ri^2 is

another quantity unaltered by the change of units, so that if the new unit of i is v times the old, the new unit of R must be $1/v^2$ times the old unit.

Dimensions of Electrical Quantities.

256. For the general theory of Dimensions we shall refer the reader to Maxwell's *Theory of Heat*, Chap. IV.; we shall in this chapter confine our attention to the dimensions of electrical quantities.

It may be well to state at the outset that the 'dimensions' of electrical quantities are a matter of definition and depend entirely upon the system of units we adopt. Thus we shall find that on the electromagnetic system of units a resistance has the same dimensions as a velocity, while on the electrostatic system of units it has the same dimensions as the reciprocal of a velocity. In fact we might choose a system of units so as to make any one electrical quantity of any assigned dimensions; when the dimensions of this are fixed that of the others becomes quite determinate.

A symbol representing an electrical quantity merely tells us how much of the quantity there is, and does not tell us anything about the nature of the quantity; this would require a dynamical theory of electricity. A theory of dimensions cannot tell us what electricity is; its object is merely to enable us to find the change in the numerical measure of a given charge of electricity or any other electrical quantity when the units of length, mass and time are changed in any determinate way.

We have to fix the electrical quantities by one or other of their properties. Thus, to take an example, we may fix a charge of electricity by the repulsion it exerts on an equal charge, as is done in the electrostatic system of units, or by the force experienced by a magnetic pole when the charge is being transferred from one place to another by a current, as is done in the electromagnetic system; these two measures are of different dimensions. To take a simpler case we might fix a quantity of water by the number of hydrogen atoms it contains, by its mass, or by its volume at a definite temperature; all these measures would be of different dimensions.

On the electrostatic system of units the force between two equal charges e, separated by a distance L in a medium whose specific inductive capacity is K, is e^2/KL^2, and since this is of the dimensions of a force we have the dimensional equation

$$\frac{e^2}{KL^2} = \frac{ML}{T^2} \qquad \text{(1)},$$

M, L, T representing mass, length and time.

This result, with the meaning assigned to K in Art. 68, is only true on the electrostatic system of units. We may, however, generalize the meaning of K and say that whatever be the system of units, the repulsion between the charges is e^2/KL^2, where K is defined as the 'specific inductive capacity of the medium on the new system of units.' We may regard this as the definition of K on this system. The ratio of the K's for two substances on this system is of course the same as the ratio of the K's on the electrostatic system. We shall regard the dimensions of K as indeterminate and keep them in the expression for the dimensions of the electrical quantities*. From equation (1) we have the dimensional equation

$$e = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}K^{\frac{1}{2}}.$$

Similarly on the electromagnetic system of units the repulsion between two poles of strength m separated by a distance L in a medium whose magnetic permeability is μ is $m^2/\mu L^2$, μ for this system of units being a quantity of no dimensions. We shall suppose that whatever be the system of units the force between the poles is equal to $m^2/\mu L^2$: where μ thus determined is defined as the magnetic permeability of the medium on this system of units. Thus, for example, if m is the measure, on the electrostatic system of units, of the strength of a pole, the force between two equal poles separated by unit distance in air is not m^2 but $9 \times 10^{20} m^2$. Hence we say the magnetic permeability of air on the electrostatic system of units is $1/9 \times 10^{20}$. We shall regard the dimensions of μ as being left undetermined and retain μ in the expressions for the dimensions of the electric quantities. Since $m^2/\mu L^2$ is of the dimensions of a force we have the dimensional equation

$$m = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}\mu^{\frac{1}{2}}.$$

* Rücker, *Phil. Mag.* vol. 27, p. 104.

We shall find it instructive to suppose that the electric and agnetic units are connected together by the relation that p times e work done by unit pole in traversing a closed circuit is equal to · times the current flowing through the circuit: the convention ade on both the electrostatic and magnetic systems is that p is quantity of no dimensions and always equal to unity. We shall r the present leave the dimensions of p undecided.

The dimensional equation connecting the electric and magnetic antities is therefore

$$p \times H \times L = i,$$

here H is magnetic force, L a length and i a current.

Taking this relation and starting with the electric charge, we n get by the equations given in Art. 255 the dimensions of all the ectrical and magnetic quantities in terms of M, L, T, p, K: or arting with the magnetic pole we can get them in terms of M, L, , p, μ. The results for some of the more important electrical antities are given in the following table.

Quantity	Symbol	Dimensions in terms of K and p	Dimensions in terms of μ and p
harge	e	$K^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$	$p\mu^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}$
lectric intensity	F	$K^{-\frac{1}{2}}M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$	$p^{-1}\mu^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}$
otential difference	V	$K^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	$p^{-1}\mu^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$
urrent	i	$K^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$	$p\mu^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$
esistance	R	$K^{-1}L^{-1}T$	$p^{-2}\mu LT^{-1}$
lectric polarization	D	$K^{\frac{1}{2}}M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$	$p\mu^{-\frac{1}{2}}M^{\frac{1}{2}}L^{-\frac{3}{2}}$
apacity	C	KL	$p^2\mu^{-1}L^{-1}T^2$
pecific inductive capacity ...	K	K	$p^2\mu^{-1}L^{-2}T^2$
trength of Magnetic pole ...	m	$pK^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}$	$\mu^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$
Iagnetic force	H	$p^{-1}K^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}$	$\mu^{-\frac{1}{2}}M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$
Iagnetic induction	B	$pK^{-\frac{1}{2}}M^{\frac{1}{2}}L^{-\frac{3}{2}}$	$\mu^{\frac{1}{2}}M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$
Iagnetic permeability ...	μ	$p^2K^{-1}L^{-2}T^2$	μ

We see from this table that the dimensions of K, μ, and p must n all systems of measurement be connected by the relation

$$\frac{p^2}{\mu K} = \frac{L^2}{T^2} = (\text{velocity})^2.$$

On Maxwell's theory of the electric field $p/\sqrt{\mu K}$ is equal to the velocity with which electric disturbances travel through a medium whose magnetic permeability is μ and specific inductive capacity K.

On the electrostatic system of units K is of no dimensions, as the specific inductive capacity of air is taken as unity whatever may be the units of mass, length and time. Also on this system p is by hypothesis of no dimensions, being always equal to unity. Hence the dimensions of the electrical quantities on this system of units are got by omitting p and K in the third column of the table.

On the electromagnetic system of units μ is of no dimensions, the magnetic permeability of air being taken as unity whatever the units of mass, length and time; p is also of no dimensions on this system. Hence the dimensions of the electrical quantities on this system of units are got by omitting μ and p from the fourth column in the table.

Another system of units could be got by taking μ and K as of no dimensions and p a velocity. If this velocity were taken equal to the ratio of the electromagnetic unit charge to the electrostatic unit, then the unit of electric charge on this system would be the ordinary electrostatic unit of that quantity, while the unit magnetic pole would be the unit as defined on the electromagnetic system. This system would thus have the advantage that the electric quantities would be as defined in the electrostatic system, while the magnetic quantities would be as defined in the magnetic system, and we should not have to introduce any new definitions: whereas if we use the electrostatic system we have to define all the magnetic quantities afresh, and if we use the electromagnetic system we have to re-define all the electrical ones*.

* It should be noticed that it is only when the electromagnetic system of units is used that 'magnetic induction' has the meaning assigned to it in Art. 153. If we use any system of units in which we start from electrical quantities, the 'magnetic induction through unit area' appears as the quantity whose rate of variation is equal to p times the electromotive force round the boundary of the area. The magnetic induction defined in this way is always proportional to the magnetic induction as defined in Art. 153. The two are however only identical on the electromagnetic system of units. With the definition of Art. 153 the magnetic induction is of the same dimensions as magnetic force, since they are both the mechanical force on a unit pole when placed in cavities of different shapes.

This system is however never used in practice; the electro-netic system or one founded upon it is universally used in trical Engineering, and the electrostatic system is used for ial classes of investigations.

257. The units of resistance, of electromotive force, of capacity he electromagnetic system are either too large or too small to practically convenient: hence new units which are definite iples or submultiples of the electromagnetic units are employed. se units and their relation to the electromagnetic system of units n the units of length, mass and time are the centimetre, gramme second) are given in the following table.

unit of resistance is called the Ohm and is equal to 10^9 electromagnetic units.

unit of electromotive force is called the Volt and is equal to 10^8 electromagnetic units.

unit of current is called the Ampère and is equal to 10^{-1} electromagnetic units.

unit of charge is called the Coulomb and is equal to 10^{-1} electromagnetic units.

unit of capacity is called the Farad and is equal to 10^{-9} electromagnetic units.

Microfarad is equal to 10^{-15} electromagnetic units.

Ampère is the current produced by a Volt through an Ohm.

We shall now proceed to explain the methods by which the various ntities can be measured in terms of these units: when the quan- is so measured it is said to be determined in *absolute measure.*

258. Determination of a Resistance in Absolute asure. The method given in Art. 226 enables us to compare resistances, and thus to find the ratio of any resistance to that n arbitrary standard such as the resistance of a column of cury of given length and cross section when at a given tempera- . In order to make use of the electromagnetic system of units must find the number of electromagnetic units in our standard tance, or what amounts to the same thing we must be able to ify a conductor whose resistance is the electromagnetic unit of tance.

The first method we shall describe, that of the revolving coil, was suggested by Lord Kelvin, and carried out by a committee of the British Association, who were the first to measure a resistance in absolute measure. The method was also one of those used by Lord Rayleigh and Mrs Sidgwick in their determination of the Ohm.

When a coil of wire spins about a vertical axis in the earth's magnetic field, currents are generated in the coil; these currents produce a magnetic force at the centre of the coil. If a magnet is placed at the centre of the coil, this magnetic force gives rise to a couple on the magnet tending to twist the magnet in the direction in which the coil is rotating. The resistance of the coil may be deduced from the deflection of the magnet as follows.

Let H be the horizontal component of the earth's magnetic force, A the area enclosed by one turn of the coil, n the number of turns, θ the angle the plane of the coil makes with the magnetic meridian; let the coil revolve with uniform velocity ω, so that we may put

$$\theta = \omega t.$$

The number of tubes of magnetic induction passing through the coil is equal to

$$nAH \sin \theta,$$

and the rate of diminution of this is

$$-nAH\omega \cos \omega t.$$

Hence, if L is the coefficient of self-induction of the coil, R its resistance, and i the current flowing through the coil, the current being taken as positive when the lines of magnetic force due to the current and those due to the earth pass through the circuit in the same direction, we have

$$L\frac{di}{dt} + Ri = -nAH\omega \cos \omega t.$$

Hence, as in Art. 237, we have

$$i = -\frac{nAH\omega}{R^2 + \omega^2 L^2}\{R \cos \omega t + L\omega \sin \omega t\}.$$

Now if unit current through the coil produces a magnetic force G at the centre, the current i through the coil will produce a magnetic force $Gi \cos \omega t$ at right angles to the magnetic meridian, and a force

n ωt along the magnetic meridian, since $\theta = \omega t$. Hence the netic force due to the currents in the coil has a component

$$-\frac{nAHG\omega R}{2(R^2+\omega^2L^2)} - \frac{nAHG\omega}{2(R^2+\omega^2L^2)}\{R\cos 2\omega t + L\omega \sin 2\omega t\},$$

ght angles to the magnetic meridian; and a component

$$-\frac{nAHGL\omega^2}{2(R^2+\omega^2L^2)} - \frac{nAHG\omega}{2(R^2+\omega^2L^2)}\{R\sin 2\omega t - L\omega \cos 2\omega t\},$$

g the magnetic meridian.

Now suppose we have a magnet at the centre of the coil, and let moment of inertia of this magnet be so great that the time of ng is very large compared with the time of revolution of the coil. magnetic force acting on the magnet due to the current induced he coil consists, as we see, of two parts, one constant, the other odic, the frequency being twice that of the revolution of the . By making the moment of inertia of the magnet great enough may make the effect of the periodic terms as small as we please; shall suppose that the magnet is heavy enough to allow us to lect the effect of the periodic terms; when this is done the magnetic e at the centre has a component equal to

$$\frac{nAHG\omega R}{2(R^2+\omega^2L^2)}$$

ight angles to the magnetic meridian, and one equal to

$$H - \frac{nAHGL\omega^2}{2(R^2+\omega^2L^2)}$$

ng it.

Hence if ϕ is the angle the axis of the magnet at the centre of coil makes with the magnetic meridian,

$$\tan\phi = \frac{\frac{1}{2}\frac{nAHG\omega R}{R^2+\omega^2L^2}}{H - \frac{1}{2}\frac{nAHGL\omega^2}{R^2+\omega^2L^2}},$$

$$\tan\phi = \frac{\frac{1}{2}\frac{nAG\omega R}{R^2+\omega^2L^2}}{1 - \frac{1}{2}\frac{nAGL\omega^2}{R^2+\omega^2L^2}}.$$

This equation enables us to find R, as A, G, L can be calculated from the dimensions of the rotating coil. When $L\omega$ is small compared with R the equation reduces to the simple form

$$\tan\phi = \frac{1}{2}\frac{nAG\omega}{R}.$$

When the coil consists of a single ring of wire of radius a, $n = 1$, $A = \pi a^2$, $G = 2\pi/a$; hence

$$\tan\phi = \frac{\pi^2 a\omega}{R}.$$

Thus by this method we compare R, which, by Art. 256, is of the dimensions of a velocity, with the velocity of a point on the spinning coil.

The preceding investigation is only approximate as we have neglected the magnetic field due to the magnet placed at the centre of the ring.

259. Lorenz's Method. This was also one of the methods used by Lord Rayleigh and Mrs Sidgwick in their determination of the Ohm. It depends upon the principle that if a conducting disc spins in a magnetic field which is symmetrical about the axis of rotation, and if a circuit is formed by a wire, one end of which is connected to the axis of rotation while the other end presses against the rim of the disc, an electromotive force proportional to the angular velocity will act round the circuit.

We can determine this electromotive force by finding the couple acting on the disc when a current flows round this circuit.

Let I be the current flowing through the wire. When this current enters the disc at its centre it will spread out; let q be the radial current crossing unit length of the circumference of a circle of radius r at the point defined by θ. Let $rdrd\theta$ be an element of the area of the disc. The radial current flowing through this area is equal to $qrd\theta$. Hence by Art. 214, if H is the magnetic force normal to the disc at this area, the tangential mechanical force acting on the area is equal to $Hqrdrd\theta$. The moment of this force about the axis of the disc is equal to

$$Hqr^2drd\theta;$$

e the couple acting on the disc is equal to

$$\iint Hqr^2 dr d\theta,$$

ntegration being extended over the area of the disc.

Since the current flowing across a circle drawn on the disc, with entre at the centre of the disc, must equal the current I flowing the disc, we have

$$\int qr d\theta = I.$$

Since the magnetic field is symmetrical about the axis of rotation, independent of θ, hence the couple acting on the disc is equal to

$$I \int Hr dr.$$

If N be the number of tubes of magnetic induction passing ough the disc

$$N = \int H 2\pi r dr,$$

thus the couple acting on the disc is equal to

$$\frac{1}{2\pi} IN.$$

Now suppose there is a battery whose electromotive force is n the circuit, then in the time δt the work done by the battery $EI\delta t$; this work is spent in heating the circuit and in driving the . The angle turned through by the disc in this time is $\omega\delta t$, if s the angular velocity of the disc; hence the mechanical work e is equal to

$$\frac{1}{2\pi} IN\omega\delta t.$$

Joule's law the mechanical equivalent of the heat produced in circuit is equal to

$$RI^2\delta t,$$

ere R is the resistance of the circuit. Hence we have by the nservation of Energy

$$EI\delta t = RI^2\delta t + \frac{1}{2\pi} IN\omega\delta t,$$

$$I = \frac{E - \frac{1}{2\pi} N\omega}{R};$$

hence there is a counter-electromotive force in the circuit equal to

$$\frac{1}{2\pi} N\omega.$$

This case illustrates the remark made on page 292, since from Ampère's law of the mechanical force acting on currents on a magnetic field we have deduced, by the aid of the principle of the Conservation of Energy, the expression for the electromotive force due to induction, and have thus proved by dynamical principles that the induction of currents is a consequence of the mechanical force exerted by a magnet on a circuit conveying a current.

In Lord Rayleigh's experiments, the disc was placed between two coils through which a current passed, and the axes of the disc

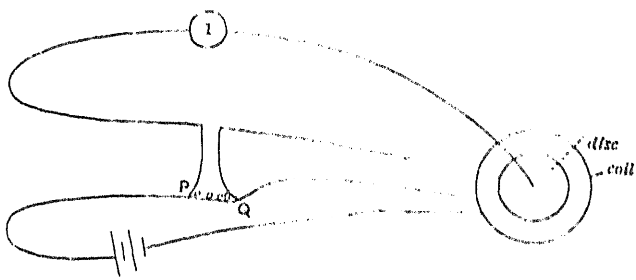


Fig. 124

and of the two coils were coincident. The magnetic field acting on the disc may be considered as approximately that due to the current through the coils, as this field is very much more intense than that due to the earth. Hence if i is the current through the coils, M the coefficient of mutual induction between the coils and a circuit coinciding with the rim of the disc,

$$N = Mi.$$

So that the electromotive force due to the rotation of the disc is

$$\frac{Mi\omega}{2\pi}.$$

The experiment was arranged as in the diagram, Fig. 124; a galvanometer was placed in the circuit connecting the centre of the disc and the rim, and this circuit was connected to two points P, Q in the circuit in series with the coils, and the resistance between P and Q was adjusted until no current passed through the galvano-

r. If R is the resistance between P and Q, and if a current i flows ugh PQ the E.M.F. between P and Q will be Ri, but, since there current through the galvanometer, this balances the electrove force due to the rotation of the disc; hence

$$Ri = \frac{Mi\omega}{2\pi},$$

$$R = \frac{M\omega}{2\pi}.$$

ince M can be calculated from the dimensions of the coil and lisc, this formula gives us R in absolute measure.

60. The method given in Art. 251 for determining a coefficient utual induction in terms of a resistance may be used to determ a resistance in absolute measure. If we use a pair of coils e coefficient of mutual induction can be determined by calculathen equation (2) of Art. 251 will give the absolute measure of sistance. This method has been employed by Sir Richard ebrook.

he result of a large number of experiments made by the preng methods is that the Ohm is the resistance at 0° C. of a column ercury 106·3 cm. long and 1 sq. millimetre in cross section.

'or a comparison of the relative advantages of the preceding ods the student is referred to a paper by Lord Rayleigh in the *osophical Magazine* for November, 1882.

261. **Absolute Measurement of a Current.** A current be determined by measuring the attraction between two coils ed in series with each other and with their planes parallel and ght angles to the line joining their centres. If i is the current ugh the coils, M the coefficient of mutual induction between the , x the distance between their centres, the attraction between oils is equal to

$$-\frac{dM}{dx}i^2.$$

By attaching one of the coils to the scale-pan of a balance and ing the other fixed we can measure this force, and hence if we late dM/dx from the dimensions of the coils we can determine absolute measure.

The unit current is very conveniently specified by the amount of silver deposited from a solution of silver nitrate through which this current has been flowing for a given time.

Lord Rayleigh found that the Ampère is the current which flowing uniformly for one second would cause the deposition of ·001118 gramme of silver.

262. The unit electromotive force is that acting on a conductor of unit resistance when conveying unit current. A practical standard of electromotive force is the Clark cell (Art. 183), whose electromotive force at $t°$ Centigrade is equal to

$$1{\cdot}434\,\{1 - {\cdot}00077\,(t - 15)\}\text{ volts.}$$

263. Ratio of Electrostatic and Electromagnetic Units. The table given on page 342 shows that the ratio of the measure of any electrical quantity on the electrostatic system of measurement to the measure of the same quantity on the electromagnetic system, is always some power of a certain quantity which we denoted by "v," and which is the ratio of the electromagnetic unit of electric charge to the electrostatic unit.

The measurement of the same electrical quantity on the two systems of units will enable us to find "v." The quantity which has most frequently been measured with this object is the capacity of a condenser. The electrostatic measure of the capacity can be calculated from the dimensions of the condenser; thus the electrostatic measure of the capacity of a sphere is equal to its radius; the capacity of two concentric spheres of radii a and b is $ab/(b-a)$; the capacity of two coaxial cylinders of length l, radii a and b, is $\frac{1}{2}l/\log b/a$. Thus if we choose a condenser of suitable shape the electrostatic measure can be calculated from its dimensions.

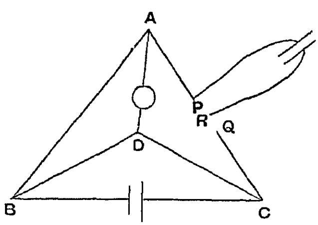


Fig. 125

The electromagnetic measure can be determined by the following method due to Maxwell. One of the arms AC of a Wheatstone's Bridge is cut at P and Q (Fig. 125), one plate of the condenser is connected to P, the other to a vibrating piece R which oscillates

kwards and forwards between P and Q; when R comes into tact with Q the condenser gets charged, when into contact with t gets discharged. The current through the galvanometer may livided into two parts. There is first a steady current which flows ough AD when no electricity is flowing into the condenser, this shall denote by $\dot{y}$. Besides this there is at times a transient rent which flows while the condenser is being charged. We shall pose that each time the condenser is being charged a quantity lectricity equal to Y flows through DA in the opposite direction $\dot{y}$. Then if the condenser is charged n times a second the amount ch flows through the galvanometer owing to the charging of the denser is nY. If the time of swing of the galvanometer needle is y long compared with $1/n$ of a second this will produce the same ct on the galvanometer as a steady current whose intensity is flowing from D to A. Thus if $nY = \dot{y}$, the current due to the eated charging of the condenser will just balance the steady rent and there will be no deflection of the galvanometer.

We now proceed to find Y. This is evidently equal to the quantity lectricity which would flow from A to D if there were no electro-tive force in the wire BC and the plates of the condenser with the atest charge they acquire in the experiment were connected to and Q respectively.

Let $\dot{Z}$ be the current from the condenser along PA during the harge, $\dot{Y}$ the current along AD, $\dot{W}$ the current along BD. Let resistances of AB, BC, CD, DB, DA be c, a, γ, β, α respectively. the coefficients of self-induction of these circuits be L_1, L_2, L_3, L_5 respectively. Then from the circuit ABD, we have

$$L_5 \frac{d^2 Y}{dt^2} + L_1 \frac{d^2 (Y - Z)}{dt^2} - L_4 \frac{d^2 W}{dt^2} + \alpha \frac{dY}{dt}$$
$$+ c\left\{\frac{dY}{dt} - \frac{dZ}{dt}\right\} - \beta \frac{dW}{dt} = 0.$$

egrating from just before discharging until after the condenser ompletely discharged, and remembering that both initially and lly $\dot{Y}$, $\dot{Z}$, $\dot{W}$ vanish, we have

$$\alpha Y + c(Y - Z) - \beta W = 0 \quad\text{.................(1)},$$

ere Y, Z, W are the quantities of electricity which have passed ing the discharge through AD, PA, and BD respectively.

Similarly from the circuit DBC, we have

$$(\beta+\gamma+a)\,W+(\gamma+a)\,Y-aZ=0 \quad \text{........(2)}.$$

We find from equations (1) and (2)

$$Y=\frac{c\,(\beta+\gamma+a)+a\beta}{\beta\,(\gamma+a)+(\alpha+c)\,(\beta+\gamma+a)}\,Z \quad \text{...........(3)}.$$

Now Z is the maximum charge in the condenser; hence if C is capacity of the condenser, and $\mathbf{A}$ and $\mathbf{C}$ the potentials of A and C respectively when the charge is a maximum, i.e. when no current is flowing into the condenser,

$$Z=C\{\mathbf{A}-\mathbf{C}\}.$$

If $\dot{y}$ is the current flowing through AD when no current is flowing in the condenser, and $\mathbf{D}$ denotes the potential of D,

$$\mathbf{A}-\mathbf{D}=\alpha\dot{y},$$

$$\mathbf{D}-\mathbf{C}=\gamma\left\{1+\frac{\alpha+c}{\beta}\right\}\dot{y},$$

$$\therefore\ \mathbf{A}-\mathbf{C}=\left\{\alpha+\gamma+\frac{\gamma}{\beta}(\alpha+c)\right\}\dot{y}.$$

Hence by equation (3)

$$Y=\frac{c\,(\beta+\gamma+a)+a\beta}{\beta\,(\gamma+a)+(\alpha+c)\,(\beta+\gamma+a)}\left\{\alpha+\gamma+\frac{\gamma}{\beta}(\alpha+c)\right\}C\dot{y}.$$

But when there is no deflection of the galvanometer

$$nY=\dot{y};$$

hence

$$\frac{1}{nC}=\frac{c\,(\beta+\gamma+a)+a\beta}{\beta\,(\gamma+a)+(\alpha+c)\,(\beta+\gamma+a)}\left\{\alpha+\gamma+\frac{\gamma}{\beta}(\alpha+c)\right\}.$$

If we know the resistances and n, we can deduce from this equation the value of C in electromagnetic measure. In practice the resistance of the battery a is very small compared with the other resistances, hence putting $a=0$, we find that approximately

$$\frac{1}{nC}=\frac{c\gamma}{\beta}\,\frac{\left\{1+\dfrac{\alpha\beta}{\gamma\,(\alpha+c+\beta)}\right\}}{1-\dfrac{\beta^2}{(\alpha+c+\beta)\,(\beta+\gamma)}}.$$

By this method we find the electromagnetic measure of the acity of a condenser; the electrostatic measure can be found from dimensions.

Now by Art. 255

$$v^2 = \frac{\text{electrostatic measure of a condenser}}{\text{electromagnetic measure of the same condenser}}.$$

Experiments made by this method show that

$$v = 3 \times 10^{10} \text{ cm./sec. very nearly.}$$

CHAPTER XIII

DIELECTRIC CURRENTS AND THE ELECTROMAGNETIC THEORY OF LIGHT

264. The Motion of Faraday Tubes. Dielectric Currents. In Chapter XI. we considered the relation between the currents in the primary and secondary circuits when an alternating current passes through the primary circuit, we did not however discuss the phenomena occurring in the dielectric between the circuits. As we regard the dielectric as the seat of the energy due to the distribution of the currents, the study of the effects in the dielectric is of primary importance. We owe to Maxwell a theory, now in its main features universally accepted, by which we are able to determine completely the electrical conditions, not merely in the conductors but also in every part of the field. We shall also see that Maxwell's views lead to a comprehensive theory of optical as well as of electrical phenomena, and enable us by means of electrical principles to explain the fundamental laws of Optics.

Before specifying in detail the principles of Maxwell's theory, we shall endeavour to show by the consideration of some simple cases that in considering the relation between the work done in taking unit magnetic pole round a closed circuit and the current flowing through that circuit (see Art. 203), we must include under the term *current*, effects other than the passage of electricity through conducting media, if we are to retain the conception that the dielectric is the seat of the energy in electric and magnetic phenomena.

Let us consider the case of a long, straight, cylindrical conductor carrying an alternating electric current. In the dielectric around this wire there is a magnetic field, and, according to the views enunciated in Art. 163, there is in a unit volume of the dielectric at a place where the magnetic force is H an amount of energy equal to $\mu H^2/8\pi$. As the alternating current changes in intensity, the energy in the surrounding field changes, and this change in the energy must be due to the motion of energy from one part of the

to another, the energy moving radially towards or away from wire conveying the current. If the dielectric medium possesses tia, and if its properties in any way resemble those of any kind natter with which we are acquainted, the energy cannot travel one place to another with an infinite velocity.

As the alternating current changes, the energy in the field will nge also; when the current is passing through its zero value, it vident that the magnetic energy cannot now vanish throughout field, for we assume that the energy travels at a finite rate, and only a finite time since the current was finite. If the magnetic gy did vanish it would imply that the energy could travel over stance, however great, in a finite time. If, however, the magnetic gy does not vanish simultaneously all over the field, there must laces where the magnetic force does not vanish. But the current ugh the conductor vanishes and there are no magnetic substances he field. Hence we conclude that unless we assume that the gy in the magnetic field can travel from one place to another an infinite velocity, we must admit that in a variable field netic forces can arise apart from magnets or electric currents ugh conductors.

265. Let us now see if we can find any clue as to what produces magnetic field under these circumstances. Let us consider the wing simple case. Let A, B (Fig. 126) be two vertical al plates forming a parallel plate condenser, and let upper ends of these plates be connected by a wire of resistance. Suppose that initially the plate A is rged with a uniform distribution of positive electricity le B is charged with an equal distribution of negative tricity. If the plates are disconnected, horizontal aday tubes at rest will stretch from one plate to the er. When the plates are connected by the wire the zontal Faraday tubes will move vertically upwards towards wire. Let v be the velocity of these tubes, and σ the surface sity of the electricity on the plates, then the upward current sing across unit length in the plate A and the downward rent in B are equal to $v\sigma$. By Art. 209 these currents produce a uniform magnetic field between the plates, the

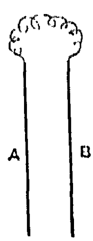


Fig. 126

magnetic force being at right angles to the plane of the paper and its magnitude equal to $4\pi v\sigma$. If N is the number of Faraday tubes passing through unit area of a plane in the dielectric parallel to the plates of the condenser $N = \sigma$. Thus the magnetic force between the planes is equal to $4\pi N v$. The condition of things between the plates is such that we have the Faraday tubes moving at right angles to themselves, and that we have also a magnetic force at right angles both to the Faraday tubes and to the direction in which they are moving; while the intensity of this force is equal to 4π times the product of the number of tubes passing through unit area and the velocity of these tubes.

Let us now see what are the consequences of generalizing this result, and of supposing that the relation between the magnetic force

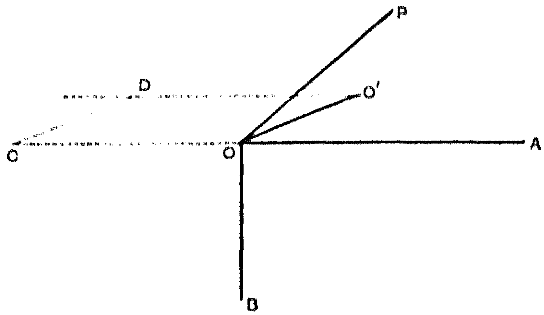


Fig. 127

and the Faraday tubes which exists in this simple case is generally applicable to all magnetic fields. Suppose then that whenever we have movements of the Faraday tubes we have magnetic force and conversely, and that the relation between the magnetic force and the Faraday tubes is that the magnetic force is equal to 4π times the product of the 'polarization' (Art. 70) and the velocity of the Faraday tubes at right angles to the direction of polarization; and that the direction of the magnetic force is at right angles to both the direction of polarization and the direction in which the Faraday tubes are moving.

We shall begin by considering what on this view is the physical meaning of $H' \times OO'$, where OO' is a line so short that the magnetic force may be regarded as constant along its length, and H' is the component of the magnetic force along OO'.

Let OA (Fig. 127) represent in magnitude and direction the ocity of the Faraday tubes, and OP the polarization; then if B represents the magnetic force, OB will be at right angles to A and OP and equal to

$$4\pi \,.\, OA \,.\, OP \sin\phi,$$

ere ϕ is the angle POA. The component H' of the magnetic force ong OO' will be

$$4\pi \,.\, OA \,.\, OP \sin\phi \cos\theta,$$

ere θ is the angle BOO'. Thus we have

$$H' \times OO' = 4\pi \,.\, OA \,.\, OP \,.\, OO' \sin\phi \cos\theta$$

$$= 24\pi\Delta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

ere Δ is the volume of the tetrahedron three of whose sides are A, OP, OO'.

Let us now find the number of Faraday tubes which cross OO' unit time. To do this, draw OC and $O'D$ equal and parallel to O, OA being the velocity of the Faraday tubes. Then the number tubes which cross OO' in unit time is the number of tubes passing rough the area $OCDO'$.

The area of the parallelogram $OCDO'$ is equal to

$$OA \times OO' \sin AOO'.$$

The number of tubes passing through it is therefore

$$OP \times \sin\theta' \times OA \times OO' \sin AOO' \quad \ldots\ldots\ldots\ldots(2),$$

ere θ' is the angle between OP and the plane of the parallelogram CDO'; this is the same as the angle between OP and the plane OO'. But

$$6\Delta = OP \times \sin\theta' \times OA \times OO' \sin AOO',$$

ere Δ as before is the volume of the tetrahedron $POO'A$. Hence om (1) and (2) we see that

$$' \times OO' = 4\pi \text{ (number of Faraday tubes crossing } OO' \text{ in unit time).}$$

Thus $\int H'ds$ where the integral is taken round a closed curve is ual to 4π times the number of tubes which pass inwards across e curve in unit time.

In Art. 203 $\int H'ds$ was taken as equal to 4π times the currents

flowing through the space enclosed by the curve, and the only currents discussed in that article were currents flowing through conductors: we shall now consider what interpretation we must attach to the new expression we have just found for $\int H'ds$.

In the first place, any tube which in unit time passes inwards across one part of the curve and outwards across another part, will not contribute anything to the total number of tubes passing across the closed curve, for its contribution when it passes inwards is equal and opposite to its contribution when it passes outwards. Hence all the tubes we need consider are those which only cross the curve once, which pass inwards across the curve and do not leave it within

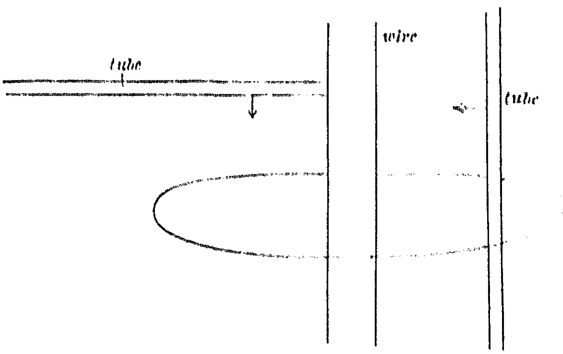


Fig. 128

unit time. These tubes may be divided into two classes, (1) those which remain within the curve, (2) those which manage to disappear without again crossing the boundary. The first set will increase the total polarization over any closed surface bounded by the curve, and the number of those which cross the boundary in unit time is equal to the rate of increase in this total polarization. The existence of the second class of tubes depends upon the presence of conductors threading the circuit, or of charged bodies moving through the area bounded by the curve. Thus suppose we have a metal wire passing through the circuit, then the tubes which cross the boundary may run into this wire and be annulled, the disappearance of each unit tube corresponding to the passage of unit electricity along the wire; or a tube might have one end on the wire and cross the circuit, its end running

ong the wire; the passage of such a tube across the boundary eans the passage of a unit of electricity along the wire; again, one d of a tube might be on a charged body which moves through the rcuit. Thus the number of tubes of class (2) which cross the circuit unit time is equal to the number of units of electricity which pass that time along conductors or on charged bodies passing through e circuit, i.e. it is equal to the sum of the conduction and convection rrents flowing through the circuit.

Hence the work done when unit pole is taken round a closed rcuit is equal to 4π times the sum of the conduction and convection rrents flowing through that circuit plus the rate of increase of the tal polarization through the circuit. From this we see that a change the polarization through the circuit produces the same magnetic fect as a conduction current whose intensity is equal to the rate of crease of the polarization. We shall call the rate of increase in the larization the *dielectric current*. The recognition of the magnetic fects due to these dielectric currents is the fundamental feature of axwell's Theory of the Electric Field. We have given a method regarding the magnetic field which leads us to expect the magnetic fects of dielectric currents. It must be remembered, however, axwell's Theory consists in the expression of this result and is not mited to any particular method of explaining it.

266. Propagation of Electromagnetic Disturbances. Ve shall now proceed to show that Maxwell's Theory leads to the onclusion that an electric disturbance is propagated through air ith the velocity of light.

We can employ the equations we deduced in Art. 234, if we egard u, v, w the components of the current, as the components of he sum of the dielectric, convection, and conduction currents. If X, Y, Z are the components of the electric intensity, and K its pecific inductive capacity, then the x, y, z components of the olarization are respectively

$$\frac{K}{4\pi}X, \quad \frac{K}{4\pi}Y, \quad \frac{K}{4\pi}Z,$$

he components of the dielectric currents are therefore

$$\frac{K}{4\pi}\frac{dX}{dt}, \quad \frac{K}{4\pi}\frac{dY}{dt}, \quad \frac{K}{4\pi}\frac{dZ}{dt}.$$

If σ is the specific resistance of the medium, the components of the conduction current are

$$\frac{X}{\sigma}, \quad \frac{Y}{\sigma}, \quad \frac{Z}{\sigma}.$$

Hence u, v, w the components of the total effective current are given by the equations

$$u = \frac{K}{4\pi}\frac{dX}{dt} + \frac{X}{\sigma},$$

$$v = \frac{K}{4\pi}\frac{dY}{dt} + \frac{Y}{\sigma},$$

$$w = \frac{K}{4\pi}\frac{dZ}{dt} + \frac{Z}{\sigma}.$$

Hence substituting these values of u, v, w in the equations of Art. 234, we get, using the notation of that Article, the following equations as the expression of Maxwell's Theory,

$$4\pi\left\{\frac{K}{4\pi}\frac{dX}{dt} + \frac{X}{\sigma}\right\} = \frac{d\gamma}{dy} - \frac{d\beta}{dz},$$

$$4\pi\left\{\frac{K}{4\pi}\frac{dY}{dt} + \frac{Y}{\sigma}\right\} = \frac{d\alpha}{dz} - \frac{d\gamma}{dx},$$

$$4\pi\left\{\frac{K}{4\pi}\frac{dZ}{dt} + \frac{Z}{\sigma}\right\} = \frac{d\beta}{dx} - \frac{d\alpha}{dy},$$

$$-\frac{da}{dt} = \frac{dZ}{dy} - \frac{dY}{dz},$$

$$-\frac{db}{dt} = \frac{dX}{dz} - \frac{dZ}{dx},$$

$$-\frac{dc}{dt} = \frac{dY}{dx} - \frac{dX}{dy}.$$

Let us now consider the case of a dielectric for which σ is infinite, so that all the currents are dielectric currents; putting σ infinite in the preceding equations, and $a = \mu\alpha$, $b = \mu\beta$, $c = \mu\gamma$, we get

$$\left.\begin{aligned} K\frac{dX}{dt} &= \frac{d\gamma}{dy} - \frac{d\beta}{dz} \\ K\frac{dY}{dt} &= \frac{d\alpha}{dz} - \frac{d\gamma}{dx} \\ K\frac{dZ}{dt} &= \frac{d\beta}{dx} - \frac{d\alpha}{dy} \end{aligned}\right\} \quad \text{..........................(1)},$$

$$\left.\begin{aligned} -\mu\frac{d\alpha}{dt} &= \frac{dZ}{dy} - \frac{dY}{dz} \\ -\mu\frac{d\beta}{dt} &= \frac{dX}{dz} - \frac{dZ}{dx} \\ -\mu\frac{d\gamma}{dt} &= \frac{dY}{dx} - \frac{dX}{dy} \end{aligned}\right\} \quad \text{..................(2).}$$

Differentiating the first equation in (1) with respect to t, we get

$$K\frac{d^2X}{dt^2} = \frac{d}{dy}\frac{d\gamma}{dt} - \frac{d}{dz}\frac{d\beta}{dt}.$$

Substituting the values of $d\gamma/dt$, $d\beta/dt$, and noticing that by (1)

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}$$

is independent of the time, we get

$$\mu K\frac{d^2X}{dt^2} = \frac{d^2X}{dx^2} + \frac{d^2X}{dy^2} + \frac{d^2X}{dz^2} \quad \text{..................(3).}$$

We may by a similar process get equations of the same form for Y, Z, a, b, c.

To interpret these equations let us take the simple case when the quantities are independent of the coordinates x, y. Equation (3) then takes the form

$$\mu K\frac{d^2X}{dt^2} = \frac{d^2X}{dz^2} \quad \text{..........................(4).}$$

If we put

$$\xi = z - \frac{t}{\sqrt{\mu K}},$$

$$\eta = z + \frac{t}{\sqrt{\mu K}},$$

and change the variables from z and t to ξ and η, we get

$$\frac{d^2X}{d\xi\, d\eta} = 0.$$

The solution of which is

$$\begin{aligned} X &= F(\xi) + f(\eta) \\ &= F\left(z - \frac{t}{\sqrt{\mu K}}\right) + f\left(z + \frac{t}{\sqrt{\mu K}}\right) \quad \text{..............(5),} \end{aligned}$$

where F and f denote any arbitrary functions.

Since $F(z - t/\sqrt{\mu K})$ remains constant as long as $z - t/\sqrt{\mu K}$ is constant, we see that if a point travels along the axis of z in the positive direction with the velocity $1/\sqrt{\mu K}$, the value of $F(z - t/\sqrt{\mu K})$ will be constant at this point. Hence the first term in equation (5) represents a value of X travelling in the positive direction of the axis of z with the velocity $1/\sqrt{\mu K}$. Similarly the second term in (5) represents a value of X travelling in the negative direction along the axis of z with the velocity $1/\sqrt{\mu K}$. For example, suppose that when $t = 0$, X is zero except between $z = +\epsilon$, $z = -\epsilon$ where it is equal to unity, and suppose further that dX/dt is everywhere zero when $t = 0$. Then equation (5) shows that after a time t

$$X = \frac{1}{2} \text{ between } z = \frac{t}{\sqrt{\mu K}} - \epsilon, \text{ and } z = \frac{t}{\sqrt{\mu K}} + \epsilon,$$

and between $z = -\dfrac{t}{\sqrt{\mu K}} - \epsilon$, and $z = -\dfrac{t}{\sqrt{\mu K}} + \epsilon$,

and is zero everywhere else. Thus the quantity represented by X travels through the dielectric with the velocity $1/\sqrt{\mu K}$.

It is shown in treatises on Differential Equations that equation (3), the general form of the equation (4), represents a disturbance travelling with the velocity $1/\sqrt{\mu K}$.

Thus Maxwell's Theory leads to the result that electric and magnetic effects are propagated through the dielectric with the velocity $1/\sqrt{\mu K}$.

Let us see what this velocity is when the dielectric is air. Using the electromagnetic system of units we have for air $\mu = 1$, $K = \dfrac{1}{v^2}$, where v is the ratio of the electromagnetic unit of electricity to the electrostatic unit (Art. 255). Hence on Maxwell's Theory electric and magnetic effects are propagated through air with the velocity "v." Now experiments made by the method described in Art. 263 lead to the result that, within the errors of experiment, v is equal to the velocity of light through air. Hence we conclude that electromagnetic effects are propagated through air with the velocity of light. This result led Maxwell to the view that since light travels with the same velocity as an electromagnetic disturbance, it is itself an electromagnetic phenomenon; a wave of light being a wave of electric and magnetic disturbances.

267. Plane Electromagnetic Waves. Let us consider e in detail the theory of a plane electric wave. If f, g, h are the ponents of the electric polarization in such a wave, l, m, n the ction cosines of the normal to the wave front, and λ the wave th, then we may put

$$f = f_0 \cos \frac{2\pi}{\lambda}(lx + my + nz - Vt),$$

$$g = g_0 \cos \frac{2\pi}{\lambda}(lx + my + nz - Vt),$$

$$h = h_0 \cos \frac{2\pi}{\lambda}(lx + my + nz - Vt);$$

re V is the velocity of propagation of the wave, and f_0, g_0, h_0 ntities independent of x, y, z or t. Since

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0,$$

have $$lf_0 + mg_0 + nh_0 = 0,$$

therefore $$lf + mg + nh = 0.$$

Thus the electric polarization is perpendicular to the direction of pagation of the wave.

By equation (2), Art. 266, we have

$$-\mu \frac{d\alpha}{dt} = \frac{dZ}{dy} - \frac{dY}{dz},$$

l $$Z = \frac{4\pi}{K} h, \quad Y = \frac{4\pi}{K} g.$$

Hence

$$\frac{d\alpha}{dt} = \frac{4\pi}{\mu K}\frac{2\pi}{\lambda}\{mh_0 - ng_0\} \sin \frac{2\pi}{\lambda}(lx + my + nz - Vt),$$

$$\alpha = \frac{4\pi}{\mu K V}(ng_0 - mh_0) \cos \frac{2\pi}{\lambda}(lx + my + nz - Vt);$$

since $$\mu K = \frac{1}{V^2},$$

$$\alpha = 4\pi V\,(ng - mh);$$

nilarly $$\beta = 4\pi V\,(lh - nf),$$

$$\gamma = 4\pi V\,(mf - lg).$$

nce $$l\alpha + m\beta + n\gamma = 0,$$

so that the magnetic force is at right angles to the direction of propagation of the wave, and since

$$f\alpha + g\beta + h\gamma = 0,$$

the magnetic force is perpendicular also to the electric polarization.

Since $\qquad \{\alpha^2 + \beta^2 + \gamma^2\}^{\frac{1}{2}} = 4\pi V\{f^2 + g^2 + h^2\}^{\frac{1}{2}},$

the resultant magnetic force is $4\pi V$ times the resultant electric polarization.

Hence in a plane electric wave, and therefore on Maxwell's Theory in a plane wave of light, there is in the front of the wave an electric polarization, and at right angles to this, and also in the wave front, there is a magnetic force bearing a constant ratio to the polarization. We shall see in Art. 270 that in a plane polarized light wave the electric polarization is at right angles to, and the magnetic force in, the plane of polarization.

In strong sunlight about $\cdot 147 \times 10^7$ ergs fall on a square centimetre per second, hence the energy per c.c. in the light is

$$\cdot 147 \times 10^7/3 \times 10^{10} = \cdot 49 \times 10^{-6} \text{ ergs}.$$

If the energy is distributed uniformly this is equal to $X^2/8\pi$ where X is the maximum electric intensity. It follows from this that X would be about 10 volts per centimetre, and the maximum magnetic force about one-fifth of the horizontal magnetic force due to the earth in England.

268. Propagation by the Motion of Faraday Tubes. The results obtained by the preceding analysis follow very simply from the view that the magnetic force is due to the motion of the Faraday tubes. The electromotive force round a circuit moving in a magnetic field is equal to the rate of diminution of the number of tubes of magnetic induction passing through the circuit. Thus let P, Q (Fig. 129) be two adjacent points on a circuit, P', Q' the positions of these points after the lapse of a time δt. Then the diminution in the time δt of the number of tubes of magnetic induction passing through the circuit of which PQ forms a part may, as in Art. 136, be shown to be equal to the number of tubes which pass through the sum of the areas $PP'Q'Q$. The number passing through $PP'Q'Q$ is equal to

$$PQ \times PP' \times B \sin\phi \sin\theta,$$

e B is the magnetic induction, ϕ the angle it makes with the $PP'Q'Q$, and θ the angle between PP' and PQ. If V is the ity with which the circuit is moving $PP' = V\delta t$. Thus the rate minution in the number of tubes passing through the circuit is

$$\Sigma PQ \,.\, VB \sin\phi \sin\theta.$$

Hence we may regard the electromotive force round the circuit quivalent to an electric intensity at each point P of the circuit se component along PQ is equal to $VB \sin\phi \sin\theta$. As the coment of this intensity parallel to B and V vanishes, the resultant nsity is at right angles to B and V and equal to

$$BV \sin\psi,$$

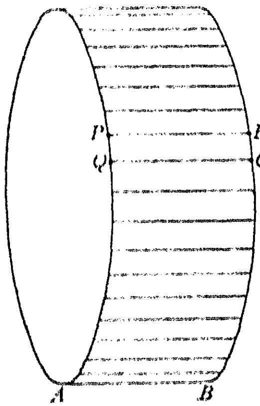


Fig. 129

re ψ is the angle between B and V. In this case the circuit was posed to move, the tubes of induction being at rest; we shall me that the same expression holds when the circuit is at rest the tubes of magnetic induction move with the velocity V across element of the circuit at rest.

Let us now introduce the view that the magnetic force is due to motion of the Faraday tubes. Let OA (Fig. 130) represent the ocity of the Faraday tubes, OP the electric polarization, and OB magnetic induction, which in a non crystalline medium is parallel he magnetic force and therefore (see page 360) at right angles OP and OA. By what we have just proved the electric intensity t right angles to OB and OA, and therefore along OC. Now in

a non-crystalline medium the electric intensity is parallel to the electric polarization; hence OP and OC must coincide in direction; thus the Faraday tubes move at right angles to their length.

Again, if E is the electric intensity, by what we have just proved

$$E = BV \quad \text{.............................}(1).$$

But if H is the magnetic force, μ the magnetic permeability,

$$B = \mu H,$$

and by Art. 265

$$H = 4\pi VP \quad \text{..........................}(2),$$

where P is the electric polarization.

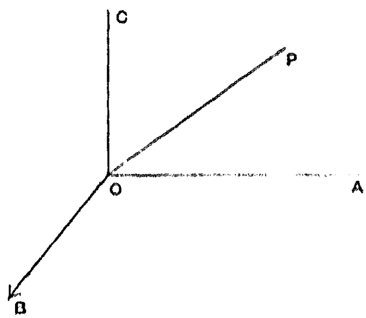


Fig. 130

Hence by (1) and (2)

$$E = 4\pi\mu V^2 P.$$

If K is the specific inductive capacity of the dielectric

$$P = \frac{K}{4\pi} E;$$

hence we have $V^2 = 1/\mu K$. The tubes therefore move with the velocity $1/\sqrt{\mu K}$ at right angles to their length.

269. Evidence for Maxwell's Theory. We shall now consider the evidence furnished by experiment as to the truth of Maxwell's theory.

We have already seen that Maxwell's theory agrees with facts as far as the velocity of propagation through *air* is concerned. We now consider the case of other dielectrics.

he velocity of light through a non-magnetic dielectric whose specific ctive capacity is K is on Maxwell's theory equal to $1/\sqrt{K}$. ence

$$\frac{\text{velocity of light in this dielectric}}{\text{velocity of light in air}} = \sqrt{\frac{\text{specific inductive capacity of air}}{\text{specific inductive capacity of dielectric}}}.$$

by the theory of light this is also equal to

$$\frac{1}{n},$$

e n is the refractive index of the dielectric. Hence on Maxwell's ry

n^2 = electrostatic measure of the specific inductive capacity.

n comparing the values of n^2 and K we have to remember that electrical conditions under which these quantities are on well's theory equal to one another, are those which hold in ave of light where the electric intensity is reversed millions of ons of times per second. We have at present no means of ctly measuring K under these conditions.

To make a fair comparison between n^2 and K we ought to take value of K determined for electrical oscillations of the same uency as those of the vibrations of the light for which n is sured. As we cannot find K for vibrations as rapid as those of visible rays, the other alternative is to use the value of n for es of very great wave length; we shall call this value n_∞.

The process by which n_∞ is obtained is not however very satis- ory. Cauchy has given the formula

$$n = A + B\lambda^{-2} + C\lambda^{-4}$$

necting n with the wave length λ, which holds accurately within limits of the visible spectrum, unless the refracting substance ne which shows the phenomenon known as 'anomalous disper- .' To find n_∞ we apply this empirical formula to determine the active index for waves millions of times the length of those used determine the constants A, B, C which occur in the formula. these reasons we should expect to find cases in which K is not al to n_∞^2, but though these cases are numerous there are many

others in which K is approximately equal to n_∞^2. A list of these is given in the following table:

Name of Substance	K	n_∞^2
Paraffin	2·29	2·022
Petroleum spirit	1·92	1·922
Petroleum oil	2·07	2·075
Ozokerite	2·13	2·086
Benzene	2·38	2·2614*
Carbon bisulphide ...	2·67	2·678*

....

As examples where the relation does not hold, we have

Glass (extra dense flint) ...	10·1	2·924*
Calcite (along axis) ...	7·5	2·197*
Quartz (along optic axis)	4·55	2·41*
Distilled water	76	1·779*

....

Sir James Dewar and Professor Fleming have shown that the abnormally high specific inductive capacities of liquids such as water, disappear at very low temperatures, the specific inductive capacities at such temperatures becoming comparable with the square of the refractive index.

Maxwell's Theory of Light has been developed to a considerable extent and the consequences are found to agree well with experiment. In fact the electromagnetic is the only theory of light yet advanced in which the difficulties of reconciling theory with experiment do not seem insuperable.

270. Hertz's Experiments. The experiments made by Hertz on the properties of electric waves, on their reflection, refraction, and polarization, furnish perhaps the most striking evidence in support of Maxwell's theory, as it follows from these experiments that the properties of these electric waves are entirely analogous to those of light waves. We regret that we have only space for an exceedingly brief account of a few of Hertz's beautiful experiments; for a fuller description of these and other experiments on electric waves with their bearings on Maxwell's theory, we refer the reader

* These are the values of n_0^2 where n_0 is the refractive index for sodium light.

ertz's own account in *Electrical Waves*, to *Recent Researches in ricity and Magnetism* by J. J. Thomson and to Fleming, *romagnetic Waves*.

'e saw in Art. 245 that when a condenser is discharged by ecting its coatings by a conductor, electrical oscillations are uced, the period of which is approximately $2\pi\sqrt{LC}$ where C is apacity of the condenser, and L the coefficient of self-induction e circuit connecting its plates. This vibrating electrical system on Maxwell's theory, be the origin of electrical waves, which l through the dielectric with the velocity V and whose wave h is $2\pi V\sqrt{LC}$. By using condensers of small capacity whose s were connected by very short conductors Hertz was able to lectrical waves less than a metre long. This vibrating electrical m is called a vibrator.

lertz used several forms of vibrators; the one used in the experi- , we are about to describe consists of two equal brass cylinders d so that their axes are coincident. The two cylinders are ected to the two terminals of an induction coil. When this is in n sparks pass between the cylinders. The cylinders correspond ie plates of the condenser, and the air between the cylinders se electric strength breaks down when the spark passes) to the uctor connecting the plates. The length of each of these cylinders out 12 cm., and their diameters about 3 cm.; their sparking are well polished.

'o detect the presence of the electrical waves, Hertz used a very ly closed metallic circuit, such as a piece of wire, bent into a e, the ends of the wire being exceedingly close together. When electric waves strike against this detector very minute sparks between the terminals; these sparks serve to detect the ence of the waves. Sir Oliver Lodge introduced a still more tive detector, founded on the fact discovered by Branly that electrical resistance of a number of metal turnings, placed so be loosely in contact with each other, is greatly affected by impact of electric waves, and that all that is necessary to detect e waves is to take a glass tube, fill it loosely with iron turnings, place the tube in series with a battery and a galvanometer. n the waves fall on the tube its resistance, and therefore the ection of the galvanometer, is altered. These earlier detectors

have now been superseded by detectors depending on the unilateral conductivity of some crystals when pressed against a conductor, and especially by hot wire valves.

The analogy between the electrical waves and light waves is very strikingly shown by Hertz's experiments with parabolic mirrors.

If the vibrator is placed in the focal line of a parabolic cylinder, and if the Faraday tubes emitted by it are parallel to this focal line; then if the laws of reflection of these electric waves are the same as for light waves, the waves emitted by the vibrator will, after reflection from the cylinder, emerge as a parallel beam, and will therefore not diminish in intensity as they recede from the

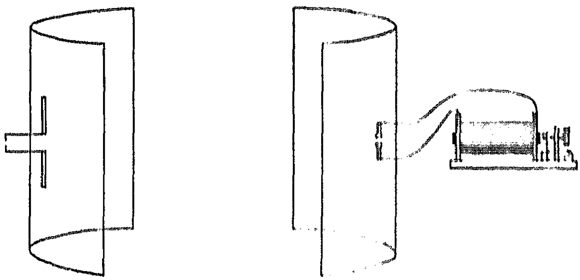


Fig. 131

mirror. When such a beam falls on another parabolic cylinder, the axis of whose cross section coincides with the axis of the beam, it will be brought to a focus on the focal line of the second mirror.

The parabolic mirrors used by Hertz were made of sheet zinc, and their focal length was about 12·5 cm. The vibrator was placed so that the axes of the cylinders coincided with the focal line of one of the mirrors. The detector, which was placed in the focal line of an equal parabolic mirror, consisted of two pieces of wire; each of these wires had a straight piece about 50 cm. long, and was then bent at right angles so as to pass through the back of the mirror, the length of the bent piece being about 15 cm. The ends of the two pieces coming through the mirror were bent so as to be exceedingly near to each other. The sparks passing between these ends were observed from behind the mirror. The mirrors are represented in Fig. 131.

Reflection of Electric Waves.

o show the reflection of these waves the mirrors were placed by side so that their openings looked in the same direction and axes converged at a point distant about 3 metres from the ors. No sparks passed between the points of the detector when ibrator was in action. If however a metal plate about 2 metres re was placed at the intersection of the axes of the mirrors, at right angles to the line which bisects the angle between the sparks appeared at the detector. These sparks however disared if the metal plate was turned through a small angle. This riment shows that the electric waves are reflected and that, oximately at any rate, the angle of incidence is equal to the of reflection.

Refraction of Electric Waves.

o show the refraction of these waves Hertz used a large prism of pitch. This was about 1·5 metres high, and it had a refracting of 30° and a slant side of 1·2 metres. When the electric waves the mirror containing the vibrator passed through this prism, sparks in the detector were not excited when the axes of the mirrors were parallel, but sparks were produced when the axis e mirror containing the detector made a suitable angle with containing the vibrator. When the system was adjusted for mum deviation, the sparks were most vigorous in the detector the angle between the axes of the mirrors was equal to 22°. would make the refractive index of pitch for these electrical s equal to 1·69.

Electric Analogy to a plate of Tourmaline.

f a properly cut tourmaline plate is placed in the path of a e polarized beam of light incident at right angles on the plate, amount of light transmitted through the tourmaline plate nds upon its azimuth. For one particular azimuth all the light be stopped, while for an azimuth at right angles to this the imum amount of light will be transmitted.

f a screen be made by winding metal wire round a large rectlar framework so that the turns of the wire are parallel to one of sides of the frame, and if this screen be interposed between

the mirrors when they are facing each other with their axes coincident, then it will stop the sparks in the detector when the turns of the wire are parallel to the focal lines of the mirrors, and thus to the Faraday tubes proceeding from the vibrator: the sparks will however recommence if the framework is turned through a right angle so that the wires are perpendicular to the focal lines of the mirror.

If this framework is substituted for the metal plate in the experiment on the reflection of waves, the sparks will appear in the detector when the wires are parallel to the focal lines of the cylinders and will disappear when they are at right angles to them. Thus this framework reflects but does not transmit Faraday tubes parallel to the wires, while it transmits but does not reflect Faraday tubes at right angles to them. It thus behaves towards the transmitted electrical waves as a plate of tourmaline does towards light waves. By using a framework wound with exceedingly fine wires placed very close together Du Bois and Rubens have recently succeeded in polarizing in this way radiant heat, whose wave length, though greater than that of the rays of the visible spectrum, is exceedingly small compared with that of electric waves.

Angle of Polarization.

When light polarized in a plane at right angles to the plane of incidence falls upon a plate of refracting substance, and the normal to the wave front makes with the normal to the refracting surface an angle $\tan^{-1}\mu$, where μ is the refractive index, all the light is refracted and none reflected. When light is polarized in the plane of incidence some of the light is always reflected.

Trouton has obtained a similar effect with electric waves. From a wall 3 feet thick reflection was obtained when the Faraday tubes proceeding from the vibrator were perpendicular to the plane of incidence, while there was no reflection when the vibrator was turned through a right angle so that the Faraday tubes were in the plane of incidence. This proves that on the electromagnetic theory of light we must suppose that the Faraday tubes are at right angles to the plane of polarization.

A very convenient arrangement for studying the properties of electric waves is described in a paper by Professor Bose in the *Philosophical Magazine* for January 1897.

CHAPTER XIV

THERMOELECTRIC CURRENTS

71. Seebeck discovered in 1821 that if in a closed circuit of
metals the two junctions of the metals are at different tempera-
, an electric current will flow round the circuit. If, for example,
nds of an iron and of a copper wire are soldered together and
f the junctions is heated, a current of electricity will flow round
ircuit; the direction of the current is such that the current flows
the copper to the iron across the hot junction, provided the
temperature of the junctions is not greater than about 600°
grade.

he current flowing through the thermoelectric circuit represents
tain amount of energy, it heats the circuit and may be made to
echanical work. The question at once arises, what is the source
is energy? A discovery made by Peltier in 1834 gives a clue to
nswer to this question. Peltier found that when a current flows
s the junction of two metals it gives rise to an absorption or
ation of heat. If it flows across the junction in one direction
is absorbed, while if it flows in the opposite direction heat is
ated. If the current flows in the same direction as the current
e hot junction in a thermoelectric circuit of the two metals
is absorbed; if it flows in the same direction as the current at
old junction of the circuit heat is liberated.

hus, for example, heat is absorbed when a current flows across
on copper junction from the copper to the iron.

he heat liberated or absorbed is proportional to the quantity
ectricity which crosses the junction. The amount of heat
ated or absorbed when unit charge of electricity crosses the
ion is called the Peltier Effect at the temperature of the
ion.

ow suppose we place an iron copper circuit with one junction
ot chamber and the other junction in a cold chamber, a thermo-

electric current will be produced flowing from the copper to the iron in the hot chamber, and from the iron to the copper in the cold chamber.

Now by Peltier's discovery this current will give rise to an absorption of heat in the hot chamber and a liberation of heat in the cold one. Heat will be thus taken from the hot chamber and given out in the cold. In this respect the thermoelectric couple behaves like an ordinary heat engine.

272. The experiments made on thermoelectric currents are all consistent with the view that the energy of these currents is entirely derived from thermal energy, the current through the circuit causing the absorption of heat at places of high temperature and its liberation at places of lower temperature. We have no evidence that any energy is derived from any change in the molecular state of the metals caused by the passage of the current or from anything of the nature of chemical combination going on at the junction of the two metals.

Many most important results have been arrived at by treating the thermoelectric circuit as a perfectly reversible thermal engine, and applying to it the theorems which are proved in the Theory of Thermodynamics to apply to all such engines. The validity of this application may be considered as established by the agreement between the facts and the result of this theory. There are however thermal processes occurring in the thermoelectric circuit which are not reversible, i.e. which are not reversed when the direction of the current flowing through the circuit is reversed. There is the conduction of heat along the metals due to the difference of temperatures of the junctions, and there is the heating effect of the current flowing through the metal which, by Joule's law, is proportional to the square of the current and is not reversed with the current. Inasmuch as the ordinary conduction of heat is independent of the quantity of electricity passing round the circuit, and the heat produced in accordance with Joule's law is not directly proportional to this quantity, it is probable that in estimating the connection between the electromotive force of the circuit, which is the work done when unit of electricity passes round the circuit, and the thermal effects which occur in it, we may leave out of account the conduction effect

the Joule effect and treat the circuit as a reversible engine. his is the case, then, as Lord Kelvin has shown, the Peltier effect not be the only reversible thermal effect in the circuit. For let ssume for a moment that the Peltier effect is the only reversible mal effect in the circuit. Let P_1 be the Peltier effect at the junction whose absolute temperature is T_1, so that P_1 is the hanical equivalent of the heat liberated when unit of electricity ses the cold junction; let P_2 be the Peltier effect at the hot ction whose absolute temperature is T_2, so that P_2 is the hanical equivalent of the heat absorbed when unit of electricity ses the hot junction. Then since the circuit is a reversible heat-ine, we have (see Maxwell's *Theory of Heat*)

$$\frac{P_1}{T_1} = \frac{P_2}{T_2} = \frac{\text{work done when unit electricity goes round the circuit}}{T_2 - T_1}.$$

, the work done when unit of electricity goes round the circuit qual to E, the electromotive force in the circuit, and hence

$$E = (T_2 - T_1) \cdot \frac{P_1}{T_1}.$$

Thus on the supposition that the only reversible thermal effects the Peltier effects at the junctions, the electromotive force round ircuit whose cold junction is kept at a constant temperature uld be proportional to the difference between the temperatures the hot and cold junctions. Cumming, however, showed that re were circuits where, when the temperature of the hot junction raised, the electromotive force diminishes instead of increasing, il, when the hot junction is hot enough, the electromotive force reversed and the current flows round the circuit in the reverse ection. This reasoning led Lord Kelvin to suspect that besides Peltier effects at the junction there were reversible thermal effects duced when a current flows along an unequally heated conductor, l by a laborious series of experiments he succeeded in establishing existence of these effects. He found that when a current of ctricity flows along a copper wire whose temperature varies from int to point, heat is liberated at any point P when the current at flows in the direction of the flow of heat at P, i.e. when the current

is flowing from hot places to cold, while heat is absorbed at P when the current flows through it in the opposite direction. In iron, on the other hand, heat is absorbed at P when the current flows in the direction of the flow of heat at P, while heat is liberated when the current flows in the opposite direction. Thus when a current flows along an equally heated copper wire it tends to diminish the differences of temperature, while when it flows along an iron wire it tends to increase those differences. This effect produced by a current flowing along an unequally heated conductor is called the Thomson effect.

Specific Heat of Electricity.

273. The laws of the Thomson effect can be conveniently expressed in terms of a quantity introduced by Lord Kelvin and called by him the 'specific heat of the electricity in the metal.' If σ is this 'specific heat of electricity,' A and B two points in a wire, the temperatures of A and B being respectively t_1 and t_2, and the difference between t_1 and t_2 being supposed small, then σ is defined by the relation,

$$\sigma(t_1 - t_2) = \text{heat developed in } AB \text{ when unit of electricity passes through } AB \text{ from } A \text{ to } B.$$

The study of the thermoelectric properties of conductors is very much facilitated by the use of the thermoelectric diagrams introduced by Professor Tait. Before proceeding to describe them we shall enunciate two results of experiments made on thermoelectric circuits which are the foundation of the theory of these circuits.

The first of these is, that if E_1 is the electromotive force round a circuit when the temperature of the cold junction is t_0 and that of the hot junction t_1, E_2 the electromotive force round the same circuit when the temperature of the cold junction is t_1, and that of the hot junction t_2, then $E_1 + E_2$ will be the electromotive force round the circuit when the temperature of the cold junction is t_0, and that of the hot junction t_2. It follows from this result that E, the electromotive force round a circuit whose junctions are at the temperatures t_0 and t_1, is equal to

$$\int_{t_0}^{t_1} Q\,dt,$$

iere Qdt is the electromotive force round the circuit when the mperature of the cold junction is $t - \frac{1}{2}dt$, and the temperature of e hot junction is $t + \frac{1}{2}dt$. The quantity Q is called the thermo-ectric power of the circuit at the temperature t.

The second result relates to the electromotive force round circuits ade of different pairs of metals whose junctions are kept at signed temperatures. It may be stated as follows: If E_{AC} is the ectromotive force round a circuit formed of the metals A, C, E_{BC} at round a circuit formed of the metals B, C, then $E_{AC} - E_{BC}$ the electromotive force acting round the circuit formed of the etals A and B; all these circuits being supposed to work between e same limits of temperature.

274. Thermoelectric Diagrams. The thermoelectric line r any metal (A) is a curve such that the ordinate represents the ermoelectric power of a circuit that metal and some standard etal (usually lead) at a tempera-re represented by the abscissa. ie ordinate is taken positive when r a small difference of tempera-re the current flows from lead to the metal A across the hot nction.

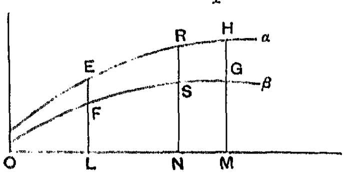


Fig. 132

It follows from Art. 273, that if the curves α and β represent the ermoelectric lines for two metals A and B, then the thermoelectric ower of a circuit made of the metals A and B at an absolute tem-erature represented by ON will be represented by RS, and the ectromotive force round a circuit formed of the two metals A and when the temperature of the cold junction is represented by OL, at of the hot junction by OM, will be represented by the area FGH.

Let us now consider a circuit of the two metals A and B with e junctions at the absolute temperatures OL_1, OL_2, Fig. 133, here OL_1 and OL_2 are nearly equal. Then the electromotive force und the circuit (i.e. the work done when unit of electrical charge asses round the circuit) is represented by the area $EHGF$. Consider ow the thermal effects in the circuit. We have Peltier effects at he junctions; suppose that the mechanical equivalent of the heat

absorbed at the hot junction when unit of electricity crosses from B to A is represented by the area P_1, let the mechanical equivalent of the heat liberated at the cold junction be represented by the area P_2. There are also the Thomson effects in the unequally heated metals; suppose that the mechanical equivalent of the heat liberated when unit of electricity flows through the metal A from the hot to the cold junction is represented by the area K_1, and that the mechanical equivalent of the heat liberated when unit of electricity flows through B from the hot to the cold junction is represented by the area K_2. Then by the First Law of Thermodynamics, we have

$$\text{area } EFGH = P_1 - P_2 + K_2 - K_1 \quad \text{...........(1)}.$$

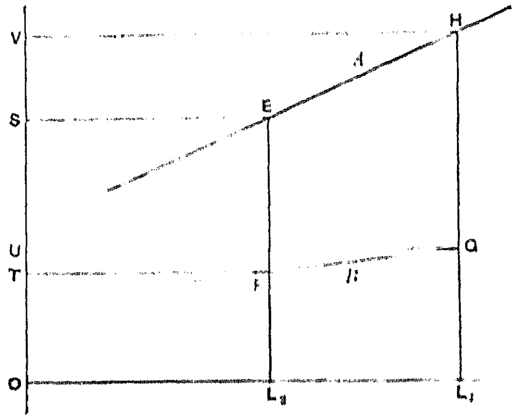


Fig. 133

The Second Law of Thermodynamics may be expressed in the form that if H be the amount of heat absorbed in any reversible engine at the absolute temperature t, then

$$\Sigma \frac{H}{t} = 0.$$

In our circuit the two junctions are at nearly the same temperature, and we may suppose that the temperature at which the absorption of heat corresponding to the Thomson effect takes place is the mean of the temperatures of the junctions, i.e. $\frac{1}{2}(OL_1 + OL_2)$.

Hence by the Second Law of Thermodynamics, we have

$$0 = \frac{P_1}{OL_1} - \frac{P_2}{OL_2} + \frac{K_2 - K_1}{\frac{1}{2}(OL_1 + OL_2)} \quad \text{..............(2)}.$$

Hence from (1) and (2) we get

$$\text{area } EFGH = \frac{1}{2}\left\{\frac{P_1}{OL_1} + \frac{P_2}{OL_2}\right\}(OL_1 - OL_2),$$

ince OL_1 is very nearly equal to OL_2 and therefore P_1 is very ly equal to P_2, this gives approximately

$$\text{area } EFGH = \frac{P_1}{OL_1}(OL_1 - OL_2).$$

But when OL_1 is very nearly equal to OL_2, the area

$$EFGH = GH\,(OL_1 - OL_2),$$

hat

$$P_1 = GH\,.\,OL_1,$$

P_1 is represented by the area $GHVU$. Now P_1 is the Peltier ct at the temperature represented by OL_1, hence we see that at temperature

eltier effect = (thermoelectric power) (absolute temperature),

$$P = Qt,$$

re t is the absolute temperature.

By the definition of Art. 273 we see that if σ_1 is the specific heat lectricity for the metal A, σ_2 that for B, then

$$K_1 - K_2 = (\sigma_1 - \sigma_2)\,L_2L_1.$$

But by (1)

$$\text{area } EFGH = P_1 - P_2 + K_2 - K_1,$$

$$P_1 = \text{area } GHVU,$$

$$P_2 = \text{area } FEST.$$

nce $\quad K_1 - K_2 = \text{area } SEHV - \text{area } TFGU$

$$= (\tan\theta_1 - \tan\theta_2)\,OL_1 \times L_2L_1,$$

re θ_1, θ_2 are the angles which the tangents at E and F to the rmoelectric lines for A and B make with the axis along which nperature is measured. Hence

$$\sigma_1 - \sigma_2 = (\tan\theta_1 - \tan\theta_2)\,OL_1 \quad \text{.................(3).}$$

When the temperature interval L_1L_2 is finite the areas $UGHV$ d $FEST$ will still represent the Peltier effects at the junctions, d the area $TFGU$ the heat absorbed when unit of electricity flows ng the metal B from a place where the temperature is OL_2 to one ere it is OL_1.

The preceding results are independent of any assumption as to the shape of the thermoelectric lines. The results of the experiments made by Professor Tait and others show, that over a considerable range of temperatures, these lines are straight for most metals and alloys, while Le Roux has shown that the 'specific heat of electricity' for lead is excessively small. Let us assume that it is zero and suppose that the diagram represents the thermoelectric lines of metals with respect to lead: then since these lines are straight, θ is constant for any metal and σ_2 vanishes when it refers to lead, the value of σ the 'specific heat of electricity' in the metal is by (3) given by the equation

$$\sigma = \tan\theta \,.\, t,$$

where t denotes the absolute temperature.

The thermoelectric power Q of the metal with respect to lead at any temperature t is given by the equation

$$Q = \tan\theta\,(t - t_0),$$

where t_0 is the absolute temperature where the line of the metal cuts the lead-line; t_0 is defined as the neutral point of the metal and lead.

Let us consider two metals; let θ_1, θ_2 be the angles their lines make with the lead-line, and t_1 and t_2 their neutral temperatures, then Q_1 and Q_2 their thermoelectric powers with respect to lead are given by the equations

$$Q_1 = \tan\theta_1\,(t - t_1),$$
$$Q_2 = \tan\theta_2\,(t - t_2);$$

hence Q, the thermoelectric power of a circuit consisting of the two metals, is given by the equation

$$Q = (\tan\theta_1 - \tan\theta_2)\,(t - T_0),$$

where T_0 is the neutral temperature for the two metals and is given by the equation

$$T_0 = \frac{t_1\tan\theta_1 - t_2\tan\theta_2}{\tan\theta_1 - \tan\theta_2}.$$

The electromotive force round a circuit formed of these metals, the temperatures of the hot and cold junctions being T_1, T_2, respectively, is equal to

$$\int_{T_2}^{T_1} Q\,dt = (\tan\theta_1 - \tan\theta_2)\,(T_1 - T_2)\,(\tfrac{1}{2}(T_1 + T_2) - T_0).$$

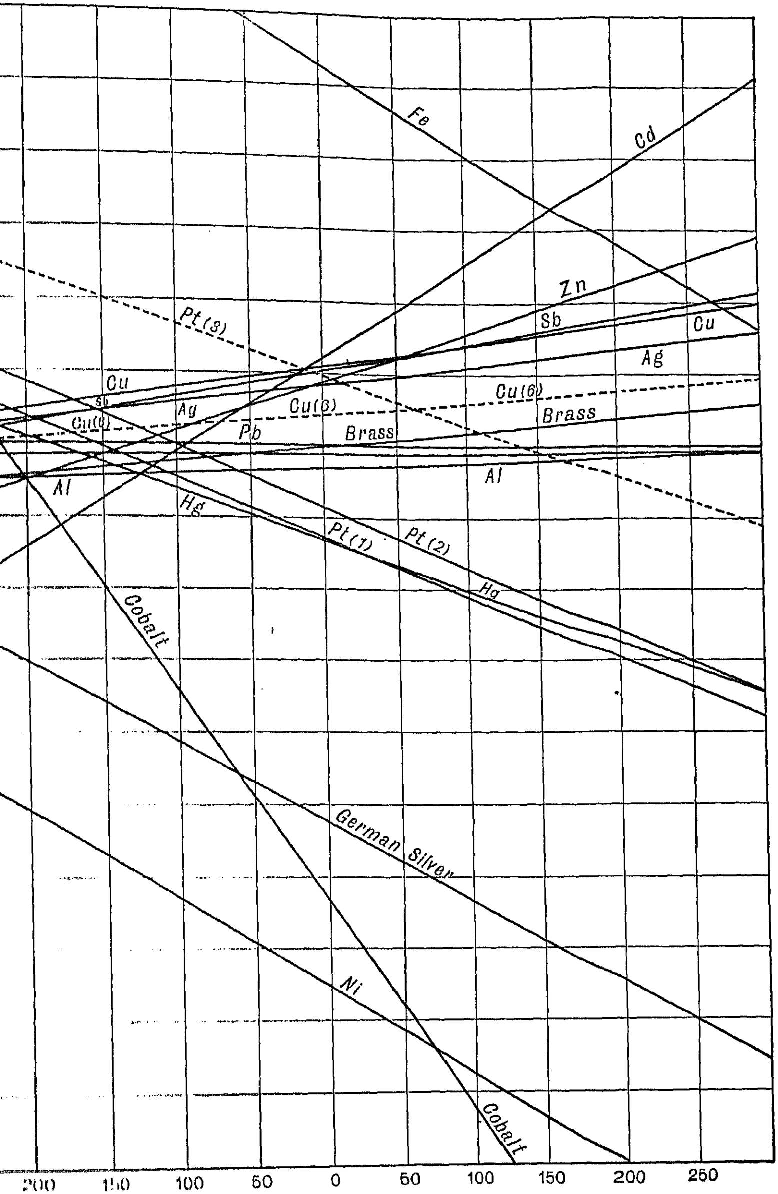

Fe
Cd
Zn
Sb
Cu
Ag
Pt (3)
Cu
Sb
Cu(6)
Ag
Cu(6)
Pb
Cu(6)
Brass
Brass
Al
Al
Hg
Pt (1)
Pt (2)
Hg
Cobalt
German Silver
Ni
Cobalt
200
150
100
50
0
50
100
150
200
250

Fig. 134

This vanishes when the mean of the temperatures of the junctions is equal to the neutral temperature. If the temperature of one junction is kept constant the electromotive force has a maximum or minimum value when the other junction is at the neutral temperature.

In Fig. 134 the thermoelectric lines for a number of metals are given. The figure is taken from a paper by Noll, *Wiedemann's Annalen*, vol. 53, p. 874. The abscissæ represent temperatures, each division being 50° C., the ordinates represent the E.M.F. for a temperature difference of 1° C., each division representing 2·5 microvolts. To find the E.M.F. round a circuit whose junctions are at t_1 and t_2 degrees we multiply the ordinate for $\frac{1}{2}(t_1 + t_2)$ degrees by $(t_2 - t_1)$.

CHAPTER XV

THE PROPERTIES OF MOVING ELECTRIC CHARGES

275. As the properties of moving electric charges are of great portance in the explanation of many physical phenomena, we all consider briefly some of the simpler properties of a moving arge and other closely allied questions.

Magnetic Force due to a Moving Charged Sphere.

The first problem we shall discuss is that of a uniformly charged here moving with uniform velocity along a straight line. Let e be e charge on the sphere, a its radius, and v its velocity; let us ppose that it is moving along the axis of z, then when things have ttled down into a steady state the sphere will carry its Faraday bes along with it. If we neglect the forces due to electromagnetic duction, the Faraday tubes will be uniformly distributed round e sphere and the number passing normally through unit area at point P will be $e/4\pi OP^2$, O being the centre of the charged sphere. hese tubes are radial and are moving with a velocity v parallel to e axis of z, hence the component of the velocity at right angles to eir direction is $v \sin\theta$, where θ is the angle OP makes with the xis of z; by Art. 265 these moving tubes will produce a magnetic rce at P equal to

$$4\pi\,(e/4\pi\,.\,OP^2)\,v\sin\theta = ev\sin\theta/OP^2.$$

he direction of this force is at right angles to the tubes, i.e. at ght angles to OP; at right angles also to their direction of motion, e. at right angles to the axis of z; thus the lines of magnetic force ill be circles whose planes are at right angles to the axis of z and hose centres lie along this axis. Thus we see that the magnetic eld outside the charged sphere is the same as that given by Ampère's ule for an element of current ids, parallel to the axis of z, placed at he centre of the sphere, provided $ev = ids$.

276. As the sphere moves, the magnetic force at P changes, so that in addition to the electrostatic forces there will be forces due to electromagnetic induction, these will be proportional to the intensity of the magnetic induction multiplied by the velocity of the lines of magnetic induction, i.e. the force due to electromagnetic induction at a point P will be proportional to $\mu\,(ev \sin\theta/OP^2) \times v$, where μ is the magnetic permeability of the medium; while the electrostatic force will be $e/K\,.\,OP^2$, where K is the specific inductive capacity of the medium. The ratio of the force due to electromagnetic induction to the electrostatic force is $\mu K v^2 \sin\theta$ or $\sin\theta v^2/V^2$, where V is the velocity of light through the medium surrounding the sphere; hence in neglecting the electromagnetic induction we are neglecting quantities of the order v^2/V^2. The direction of the force due to electromagnetic induction at P is along NP, if PN is the normal drawn from P to the axis of z; this force tends to make the Faraday tubes congregate in the plane through the centre of the sphere at right angles to its direction of motion; when the sphere is moving with the velocity of light it can be shown that all the Faraday tubes are driven into this plane.

Increase of Mass due to the Charge on the Sphere.

277. Returning to the case when the sphere is moving so slowly that we may neglect v^2/V^2; we see that since H, the magnetic force at P, is $ev\sin\theta/OP^2$, and at P there is kinetic energy equal to $\mu H^2/8\pi$ per unit volume (see Art. 163), the kinetic energy per unit volume at P is

$$\mu e^2 v^2 \sin^2\theta/8\pi\,.\,OP^4.$$

Integrating this for the volume outside the sphere, we find that the kinetic energy outside the sphere is $\frac{\mu e^2 v^2}{3a}$, where a is the radius of the sphere. Thus if m be the mass of the uncharged sphere the kinetic energy when it has a charge e is equal to

$$\frac{1}{2}\left(m + \frac{2}{3}\frac{\mu e^2}{a}\right)v^2.$$

Thus the effect of the charge is to increase the mass of the sphere by $2\mu e^2/3a$. It is instructive to compare this case with another, in which there is a similar increase in the effective mass of a body; the case we refer to is that of a body moving through a liquid.

us when a sphere moves through a liquid it behaves as if its mass re $m + \frac{1}{2}m'$, where m is the mass of the sphere, and m' the mass liquid displaced by it. Again when a cylinder moves at right gles to its axis through a liquid its apparent mass is $m + m'$, here m' is the mass of the liquid displaced by the cylinder. In e case of an elongated body like a cylinder, the increase in mass much greater when it moves sideways than when it moves point remost, indeed in the case of an infinite cylinder the increase in e latter case vanishes in comparison with that in the former; the crease in mass being $m' \sin^2 \theta$, where θ is the angle the direction motion of the cylinder makes with its axis. In the case of bodies oving through liquids the increase in mass is due to the motion the body setting in motion the liquid around it, the site of the creased mass is not the body itself but the space around it where e liquid is moving. In the electrical problem we may regard the creased mass as due to mass bound by the Faraday tubes and rried along with them as they move about. We shall for brevity eak of the source of this mass as the ether, not postulating how-er for this ether any property other than that of supplying mass r the Faraday tubes. From the expression for the energy per unit lume we see that the increase in mass is the same as if a mass $\pi\mu N^2$ per unit volume were bound by the tubes, and had a velocity ven to it equal to the velocity of the tubes at right angles to them-lves, the motion of the tubes along their length not setting this ass in motion. Thus on this view the increased mass due to the arge is the mass of ether set in motion by the tubes. If we regard oms as made up of charges of positive and negative electricity, is possible to regard all mass as electrical in its origin, and as ising from the ether set in motion by the Faraday tubes connecting e electrical charges of which the atoms are supposed to be made . For a development of this view the reader is referred to the thor's *Conduction of Electricity through Gases*; *Electricity and Matter*; d "Mass, Energy and Radiation," *Phil. Mag.*, June 1920.

Momentum in the Electric Field.

278. The view indicated above, that the Faraday tubes set the her moving at right angles to the direction of these tubes, suggests at at each point in the field there is momentum whose direction

is at right angles to the tubes, and by symmetry in the plane through the tube and the line along which the centre of the charged sphere moves. As the mass of the ether moved per unit volume at P is $4\pi\mu N^2$ where N is the density of the Faraday tubes at P, the momentum per unit volume would, on this view, be $4\pi\mu N^2 v \sin\theta$. This is equal to BN where B is the magnetic induction and N the density of the Faraday tubes at P, the direction of the momentum being at right angles to B and N. We shall now prove that this expression for the momentum is general and is not limited to the case when the field is produced by a moving charged sphere.

279. Since the magnetic force due to moving Faraday tubes is (Art. 265) equal to 4π times the density of the tubes multiplied by the components of the velocity of the tubes at right angles to their direction, and is at right angles both to the direction of the tubes and to their velocity; we see if α, β, γ are the components of the magnetic force parallel to axes of x, y, z at a place where the densities of the Faraday tubes parallel to x, y, z are f, g, h, and where u, v, w are the components of the velocity of the tubes, α, β, γ are given by the equations

$$\alpha = 4\pi(hv - gw), \quad \beta = 4\pi(fw - hu), \quad \gamma = 4\pi(gu - fv).$$

If all the tubes are not moving with the same velocity we shall have

$$\alpha = 4\pi(h_1v_1 - g_1w_1 + h_2v_2 - g_2w_2 + h_3v_3 - g_3w_3 + \ldots)$$

with similar expressions for β, γ. Here u_1, v_1, w_1 are the components of the velocity of the tubes f_1, g_1, h_1; u_2, v_2, w_2 those of the tubes f_2, g_2, h_2 and so on.

Now T the kinetic energy per unit volume at P is equal to

$$\frac{\mu}{8\pi}(\alpha^2+\beta^2+\gamma^2) = \frac{\mu}{8\pi} \times 16\pi^2 \,.\, (\{\Sigma(hv-gw)\}^2 + \{\Sigma(fw-hu)\}^2 + \{\Sigma(gu-fv)\}^2)$$
$$= 2\pi\mu\,.\,\{(\Sigma(hv-gw))^2 + (\Sigma(fw-hu))^2 + (\Sigma(gu-fv))^2\};$$

the momentum per unit volume parallel to x due to the tubes f_1, g_1, h_1 is equal to $\dfrac{dT}{du_1}$, i.e. to

$$-4\pi\mu\{h_1\Sigma(fw-hu) - g_1\Sigma(gu-fv)\}$$
$$= \mu(g_1\gamma - h_1\beta).$$

Similarly that due to the tubes f_2, g_2, h_2 is equal to

$$\mu(g_2\gamma - h_2\beta),$$

so on, thus P the total momentum parallel to x per unit volume iven by the equation

$$P = \mu(\gamma\Sigma g - \beta\Sigma h)$$
$$= \mu(\gamma g - \beta h),$$

ere f, g, h are the densities parallel to x, y, z of the whole assemblage Faraday tubes. Similarly Q, R, the components of the momentum rallel to y and z, are given respectively by the equations

$$Q = \mu(\alpha h - \gamma f),$$
$$R = \mu(\beta f - \alpha g).$$

Thus we see that the vector P, Q, R is perpendicular to the tors $\alpha, \beta, \gamma, f, g, h$, and its magnitude is $BN \sin\theta$ where B is the gnetic induction at the point, N the density of the Faraday ines and θ the angle between B and N; hence we see that each rtion of the field possesses an amount of momentum equal to the ctor product of the magnetic induction and the dielectric polariza-on.

280 Before considering the consequences of this result, it will of interest to consider the connection between the momentum d the stresses which we have supposed to exist in the field. We ve seen (Arts. 45, 46) that the electric and magnetic forces in the ld could be explained by the existence of the following stresses:

α (1) a tension $\dfrac{KR^2}{8\pi}$ along the lines of electric force;

(2) a pressure $\dfrac{KR^2}{8\pi}$ at right angles to these lines;

ere K is the specific inductive capacity, and R the electric force;

β (1) a tension $\dfrac{\mu H^2}{8\pi}$ along the lines of magnetic force;

(2) a pressure $\dfrac{\mu H^2}{8\pi}$ at right angles to these lines;

ere μ is the magnetic permeability of the medium and H the magnetic force

The force parallel to x due to the hydrostatic pressure and this tension is equal to

$$\left(-\frac{d}{dx}\frac{K(X^2+Y^2+Z^2)}{8\pi}+\frac{d}{dx}\frac{KX^2}{4\pi}\right.$$
$$\left.+\frac{d}{dy}\frac{KXY}{4\pi}+\frac{d}{dz}\frac{KXZ}{4\pi}\right)\Delta x\Delta y\Delta z;$$

when the medium is uniform, this may be written

$$\frac{K}{4\pi}\left\{Y\left(\frac{dX}{dy}-\frac{dY}{dx}\right)-Z\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)\right.$$
$$\left.+X\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)\right\}\Delta x\Delta y\Delta z.$$

Now $\qquad KX,\ KY,\ KZ = 4\pi f,\ 4\pi g,\ 4\pi h,$

and by equation (4) Art. 234,

$$\frac{dX}{dy}-\frac{dY}{dx}=\frac{dc}{dt},\quad \frac{dZ}{dx}-\frac{dX}{dz}=\frac{db}{dt},\quad \frac{dY}{dz}-\frac{dZ}{dy}=\frac{da}{dt},$$

while $\qquad K\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)=4\pi\rho;$

thus the force parallel to x due to the electric stresses may be written

$$\left(g\frac{dc}{dt}-h\frac{db}{dt}+X\rho\right)\Delta x\Delta y\Delta z.$$

In the same way the magnetic stresses may be shown to give a force parallel to x equal to

$$\frac{\mu}{4\pi}\left\{\beta\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)-\gamma\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)\right.$$
$$\left.+\alpha\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)\right\}\Delta x\Delta y\Delta z;$$

since by Art. 234

$$\frac{d\gamma}{dy}-\frac{d\beta}{dz}=4\pi\frac{df}{dt},\quad \frac{d\alpha}{dz}-\frac{d\gamma}{dx}=4\pi\frac{dg}{dt},\quad \frac{d\beta}{dx}-\frac{d\alpha}{dy}=4\pi\frac{dh}{dt},$$

and $\qquad \mu\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)=4\pi\sigma,$

where σ is the density of the magnetism, the magnetic stresses give rise to a force parallel to x equal to

$$\left(c\frac{dg}{dt}-b\frac{dh}{dt}+\alpha\sigma\right)\Delta x\Delta y\Delta z;$$

hence the system of electric and magnetic stresses together gives rise to a force parallel to x equal to

$$\left(\frac{d}{dt}(cg - bh) + X\rho + \alpha\sigma\right)\Delta x \Delta y \Delta z.$$

The terms $X\rho$ and $\alpha\sigma$ represent the forces acting on the charged bodies and the magnets in the element of volume, and are equal to the rate of increase of momentum parallel to x of these bodies, the remaining term

$$\frac{d}{dt}(cg - bh)\,\Delta x \Delta y \Delta z$$

equals the rate of increase of the x momentum in the ether in the element of volume. This agrees with our previous investigation; for we have seen (p. 391) that the momentum parallel to x per unit volume is equal to $gc - hb$.

281. A system of charged bodies, magnets, circuits carrying electric currents &c. *and the ether* forms a self-contained system subject to the laws of dynamics; in such a system, since action and reaction are equal and opposite, the whole momentum of the system must be constant in magnitude and direction, if any one part of the system gains momentum some other part or parts must lose an equal amount. If we take the incomplete system got by leaving out the ether, this is not true. Thus take the case of a charged body struck by an electric wave, the electric force in the wave acts on the body and imparts momentum to it, no other material body loses momentum, so that if we leave out of account the ether we have something in contradiction to the third law of motion. If we take into account the momentum in the ether there is no such contradiction, as the momentum in the electric waves after passing the charged body is diminished as much as the momentum of that body is increased.

282. Another interesting example of the transference of momentum from the ether to ordinary matter is afforded by the pressure exerted by electric waves, including light waves, when they fall on a slab of a substance by which they are absorbed. Take the case when the waves are advancing normally to the slab. In each unit of volume of the waves there is a momentum equal to the

product of the magnetic induction B and the dielectric polarization N; B and N are at right angles to each other, and are both in the wave front; the momentum which is at right angles to both B and N is therefore in the direction of propagation of the wave. In the wave $B = 4\pi\mu NV$, so that $BN = \frac{1}{4\pi}\frac{B^2}{\mu V}$, V being the velocity of light; B is a periodic function, and may be represented by an expression of the form $B_0 \cos(pt - nx)$, x being the direction of propagation of the wave; the mean value of B^2 is therefore $\frac{1}{2}B_0^2$. Thus the average value of the momentum per unit volume of the wave is $\frac{1}{8\pi}\frac{B_0^2}{\mu V}$, the amount of momentum that crosses unit area of the face of the absorbing substance per unit time is therefore $\frac{1}{8\pi\mu}\frac{B_0^2}{V} \cdot V$, or $\frac{1}{8\pi\mu}B_0^2$. As the wave is supposed to be absorbed by the slab no momentum leaves the slab through the ether, so that in each unit of time $\frac{B_0^2}{8\pi\mu}$ units of momentum are communicated to the slab for each unit area of its face exposed to the light: the effect on the slab is the same therefore as if the face were acted upon by a pressure $B_0^2/8\pi\mu$. It should be noticed that μ is the magnetic permeability of the dielectric through which the waves are advancing, and not of the absorbing medium.

If the slab instead of absorbing the light were to reflect it, then if the reflection were perfect each unit area of the face would in unit time be receiving $B_0^2/8\pi\mu$ units of momentum in one direction, and giving out an equal amount of momentum in the opposite direction; the effect then on the reflecting surface would be as if a pressure $2 \times B_0^2/8\pi\mu$ or $B_0^2/4\pi\mu$ were to act on the surface. This pressure of radiation as it is called was predicted on other grounds by Maxwell; it has recently been detected and measured by Lebedew and by Nichols and Hull by some very beautiful experiments.

283 If the incidence is oblique and not direct, then if the reflection is not perfect there will be a tangential force as well as a normal pressure acting on the surface. For suppose i is the angle of incidence, B_0 the maximum magnetic induction in the incident light, B_0' that in the reflected light, then across each unit of wave front in the incident light $B_0^2/8\pi\mu$ units of momentum in the direction

of the incident light pass per unit time, therefore each unit of surface receives per unit time $\cos iB_0^2/8\pi\mu$ units of momentum in the direction of the incident light, or $\cos i \sin iB_0^2/8\pi\mu$ units of momentum parallel to the reflecting surface. In consequence of reflection

$$\cos i \sin iB_0'^2/8\pi\mu$$

units of momentum in this direction leave unit area of the surface in unit time, thus in unit time

$$\cos i \sin i\,(B_0^2 - B_0'^2)/8\pi\mu$$

units of momentum parallel to the surface are communicated to the reflecting slab per unit time, so that the slab will be acted on by a tangential force of this amount. Professor Poynting succeeded in detecting this tangential force.

Since the direction of the stream of momentum is changed when light is refracted, there will be forces acting on a refracting surface, also when in consequence of varying refractivity the path of a ray of light is not straight the refracting medium will be acted upon by forces at right angles to the paths of the ray; the determination of these forces, which can easily be accomplished by the principle of the Conservation of Momentum, we shall leave as an exercise for the student.

284. We shall now proceed to illustrate the distribution of momentum in some simple cases.

Case of a Single Magnetic Pole and an Electrified Point.

Let A be the magnetic pole, B the charged point, m the strength of the pole, e the charge on the point, then at a point P the magnetic induction is m/AP^2 and is directed along AP, the dielectric polarization is $e/4\pi BP^2$ and is along BP, hence the momentum at P is

$$\frac{me \sin APB}{4\pi \,.\, AP^2 \,.\, BP^2}$$

and its direction is the line through P at right angles to the plane APB. The lines of momentum are therefore circles with their centres along AB and their planes at right angles to it, the resultant momentum in any direction evidently vanishes. There will however be a finite *moment* of momentum about AB: this we can easily show by

ration to be equal to em. Thus in this case the distribution of entum is equivalent to a moment of momentum em about AB. listribution of momentum is similar in some respects to that in spinning about AB as axis. Since the moment of momentum e ether does not depend upon the distance between A and B it not change either in magnitude or direction when A or B moves e direction of the line joining them. If however the motion of B is not along this line, the direction of the line AB and there- the direction of the axis of the moment omentum of the ether, changes. But moment of momentum of the system sting of the ether, the charge point, the pole must remain constant; hence n the momentum in the ether changes, momentum of the system consisting of ole and the charge must change so as to ensate for the change in the momentum

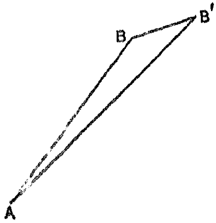


Fig. 135

e ether. Thus suppose the charged point moves from B to B' e time δt, then in that time the moment of momentum in the r changes from em along AB to em along AB'; this change in moment of momentum of the ether is equivalent to a moment omentum whose magnitude is $em\delta\theta$, where $\delta\theta = \angle BAB'$, and se axis is at right angles to AB in the plane BAB'. The change e moment of momentum of the pole and point must be equal opposite to this. Since the resultant momentum of the ether shes in any direction, the change in the momentum of the pole t be equal and opposite to the change in momentum of the t, and these two changes must have a moment of momentum l to $em\delta\theta$; we see that this will be the case if δI the change in entum of the point is at right angles to the plane BAB' and l to $\frac{em\delta\theta}{AB}$, while the change in momentum in the pole is equal opposite to this. This change in momentum $\frac{em\delta\theta}{AB}$ occurring in time δt may be regarded as produced by a force F acting on the t at right angles to the plane BAB' and given by the equation

$$F = \frac{em}{AB}\cdot\frac{\delta\theta}{\delta t}$$

Now
$$\delta\theta = \frac{BB' \sin ABB'}{AB},$$
or if v be the velocity of the point,
$$\delta\theta = \frac{v\delta t \sin ABB'}{AB},$$
or
$$\frac{\delta\theta}{dt} = \frac{v \sin ABB'}{AB};$$
thus
$$F = \frac{emv}{AB^2} \sin ABB'$$
$$= evH \sin \phi,$$
where H is the magnetic force at the point and ϕ the angle between H and the direction in which the point is moving; from this we see that a moving charged point in a magnetic field is acted on by a force at right angles to the velocity of the point, at right angles also to the magnetic force at the point, and equal to the product of the charge, the magnetic force and the velocity of the point at right angles to the magnetic force. Thus we see that we can deduce the expression for the force acting on a charged point moving across the lines of magnetic force directly from the principle of the Conservation of Momentum. We should have got an exactly similar expression if we had supposed the charge at rest and the pole in motion; in this case we must take v to be the velocity of the pole and ϕ the angle between v and AB.

285. From the expression given on page 391 for the momentum in the field we can prove that the momentum in the ether due to a charged point at P and the magnetic force produced by a current flowing round a small closed circuit, is equivalent to a momentum passing through P whose components F, G, H parallel to the axes of x, y, z respectively are given by the equations
$$F = \mu i\alpha \left(m \frac{d}{dz}\frac{1}{r} - n \frac{d}{dy}\frac{1}{r} \right),$$
$$G = \mu i\alpha \left(n \frac{d}{dx}\frac{1}{r} - l \frac{d}{dz}\frac{1}{r} \right),$$
$$H = \mu i\alpha \left(l \frac{d}{dy}\frac{1}{r} - m \frac{d}{dx}\frac{1}{r} \right),$$
where i is the current flowing round the circuit, α the area of the circuit and l, m, n the direction cosines of the normal to its plane,

x, y, z are the coordinates of P and r the distance of P from the centre of the circuit, the charge at P is supposed to be the unit charge. We see that

$$\frac{dF}{dy} - \frac{dG}{dx} = \mu i a \left\{ m \frac{d^2}{dy\,dz}\frac{1}{r} - n \left(\frac{d^2}{dy^2}\frac{1}{r} + \frac{d^2}{dx^2}\frac{1}{r} \right) + l \frac{d^2}{dx\,dz}\frac{1}{r} \right\},$$

or since

$$\frac{d^2}{dx^2}\frac{1}{r} + \frac{d^2}{dy^2}\frac{1}{r} + \frac{d^2}{dz^2}\frac{1}{r} = 0,$$

$$\frac{dF}{dy} - \frac{dG}{dx} = \mu i a \left(l \frac{d^2}{dx\,dz}\frac{1}{r} + m \frac{d^2}{dy\,dz}\frac{1}{r} + n \frac{d^2}{dz^2}\frac{1}{r} \right)$$

$$= \mu i a \frac{d}{dz} \left(l \frac{d}{dx}\frac{1}{r} + m \frac{d}{dy}\frac{1}{r} + n \frac{d}{dz}\frac{1}{r} \right)$$

$$= c,$$

c being the z component of the magnetic induction at P due to the small circuit. We have similarly if a and b are the x and y components respectively of this induction

$$\frac{dH}{dx} - \frac{dF}{dz} = b,$$

$$\frac{dG}{dz} - \frac{dH}{dy} = a.$$

The usual expression for the electromotive force due to induction follows at once from the principle of the Conservation of Momentum. For the momentum in the ether is equivalent to a momentum through P whose components are F, G, H. Suppose that in consequence of the motion of the circuit or the alteration of the current through it, F, G, H become $F + \delta F, G + \delta G, H + \delta H$, then the momentum in the ether still passes through P but has now components $F + \delta F$, $G + \delta G$, $H + \delta H$ instead of F, G, H; but the momentum of the whole system, point circuit and ether must remain constant; thus to counterbalance the changes in momentum $\delta F, \delta G, \delta H$ at P due to the ether, we must have changes in momentum of the unit charge at P equal to $-\delta F$, $-\delta G$, $-\delta H$. Suppose that the time taken by the changes $\delta F, \delta G, \delta H$ is δt, then in the time δt the x momentum of the unit charge at P must change by $-\delta F$, i.e. the unit charge must be acted on by force $-\frac{dF}{dt}$. Thus there is at P an electric force whose component parallel to x is $-\frac{dF}{dt}$, similarly the

components parallel to y and z are $-\dfrac{dG}{dt}$, $-\dfrac{dH}{dt}$. The electric force whose components we have just found is the force due to electromagnetic induction, and its magnitude is that given by Faraday's law. To prove this we notice that the line integral of the electric force round a fixed circuit of which ds is an element is equal to

$$-\int\left(\frac{dF}{dt}\frac{dx}{ds}+\frac{dG}{dt}\frac{dy}{ds}+\frac{dH}{dt}\frac{dz}{ds}\right)ds$$

$$=-\frac{d}{dt}\int\left(F\frac{dx}{ds}+G\frac{dy}{ds}+H\frac{dz}{ds}\right)ds$$

$$=\frac{d}{dt}\int\left\{l\left(\frac{dG}{dz}-\frac{dH}{dy}\right)+m\left(\frac{dH}{dx}-\frac{dF}{dz}\right)+n\left(\frac{dF}{dy}-\frac{dG}{dx}\right)\right\}dS$$

by Stokes' theorem; here l, m, n are the direction cosines of the normal to a surface filling up the closed curve, dS is an element of this surface. Substituting the values already given for $\dfrac{dG}{dz}-\dfrac{dH}{dy}$, &c. the preceding expression becomes

$$-\frac{d}{dt}\int(la+mb+nc)\,dS;$$

the integral in this expression is the number of lines of magnetic induction passing through the closed circuit, hence we see that the line integral of the electric force due to induction round a closed circuit equals the rate of diminution in the number of lines of magnetic induction passing through the circuit; this however is exactly Faraday's law of induction (see Art. 229).

286. When a charged particle is moving so rapidly that v^2/V^2 cannot be neglected, the distribution of the Faraday tubes round the particle is no longer uniform and the expression $2\mu e^2v/3a$ given in Art. 277 for the momentum of the charged sphere has to be modified.

When v approaches V, the value of momentum/v, the apparent mass, increases rapidly with v; thus if an appreciable amount of the mass of a body is due to electric charge, the mass of the body will increase with the velocity, it is only however when the velocity of the body approaches that of light that this increase becomes appreciable, in the limiting case where the velocity is that of light the apparent mass would be infinite. The influence of velocity on the apparent mass of particles travelling with great velocities has

been detected by Kaufmann by some very interesting experiments, a short account of which will be found in the author's *Conduction of Electricity through Gases*, page 533. Kaufmann found that a particle moving with a velocity about five per cent. less than the velocity of light, had a mass about three times that with small velocities.

The increase in the mass of a slowly moving charged sphere is $2\mu e^2/3a$, i.e. 4 (potential energy of the sphere)$/3V^2$, thus if this mass were to move with the velocity of light its kinetic energy would be two thirds of the electrical potential energy. The proportion between the increase in the mass due to electrification and the electrical potential energy can be shown to hold for any system of electrified bodies as well as for the simple case of the charged sphere.

287. Effects due to changes in the velocity of the moving charged body. We shall take first the case of a charged sphere moving so slowly that the lines of force are symmetrically distributed around it, and consider what will happen when the sphere is suddenly stopped. The Faraday tubes associated with the sphere have inertia and are in a state of tension, thus any disturbance communicated to one end of a tube will travel along the tube with a finite and constant velocity the velocity of light. Let us suppose that the stoppage of the particle takes a finite small time τ. We can find the configuration of the tubes, after a time t has elapsed since the sphere began to be stopped, in the following way. Describe with the centre of the charged sphere as centre two spheres, one having the radius Vt, the other the radius $V(t-\tau)$. Then since no disturbance can have reached the portions of the Faraday tubes situated outside the surface of the outer sphere these tubes will be in the positions they would have occupied if the sphere had not been stopped, while since the disturbance has passed over the tubes within the inner sphere, these tubes will be in their final position. Thus consider a tube which when the particle was stopped was along the line OPQ, O being the centre of the charged sphere, this will be the final position of the tube; hence at the time t the portion of this tube inside the inner sphere will be in the position OP, the portion $P'Q'$ outside the outer sphere will be in the position it would have occupied if the sphere had not been stopped, i.e. if O' is the position to which O would have come if the sphere had not been stopped,

$P'Q'$ will be a straight line passing through O'. Thus to preserve its continuity the tube must bend round in the shell between the surfaces of the two spheres, and take the position $OPP'Q'$. Thus the tube which before the sphere was stopped was radial, has now, in the shell, a tangential component, and this implies a tangential electric force; this tangential force is, as the following calculation shows, much greater than the radial force at P before the sphere was brought to rest.

Let us suppose that δ, the thickness of the shell, is so small that the portion of the Faraday tube inside it may be regarded as straight,

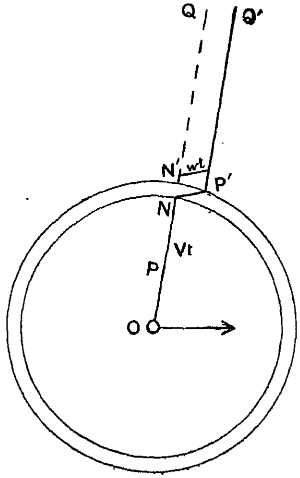


Fig. 136

then, if T is the tangential force inside the pulse, R the radial force, we have

$$\frac{T}{R} = \frac{P'N'}{PN'} = \frac{OO' \sin\theta}{\delta} = \frac{wt \sin\theta}{\delta} \quad \text{..............(1)},$$

where w is the velocity with which the sphere was moving before it was stopped, and θ the angle OP makes with the direction of motion of the sphere; t is the time since the sphere was stopped. Since $OP = Vt$ and $R = e/K \,.\, OP^2$, K being the specific inductive capacity of the medium, we have, writing r for OP,

$$T = \frac{ew \sin\theta}{KV \,.\, r\delta}.$$

Thus the tangential force varies inversely as the distance and not as the square of the distance.

The tangential Faraday tubes move radially outwards with the velocity V, they will therefore produce a magnetic force at right angles to the plane of the pulse and in the opposite direction to the magnetic force at P before the sphere was stopped; this force is equal to

$$V \times 4\pi \cdot \frac{KT}{4\pi} = \frac{ew\sin\theta}{r\delta};$$

the magnetic force before the sphere was stopped was $ew\sin\theta/r^2$, thus the magnetic force in the pulse, which however only lasts for a very short time, exceeds that in the steady field in the proportion of r to δ.

Thus the pulse produced by the stoppage of the sphere is the seat of very intense electric and magnetic forces; the pulses formed by the stoppage of the negatively electrified particles of the cathode rays form, in my opinion, the well-known Röntgen rays.

288. Energy in the Pulse. The energy due to the magnetic force in the field is per unit volume

$$\frac{\mu}{8\pi}\frac{e^2w^2\sin^2\theta}{\delta^2r^2};$$

integrating this through the pulse we find that the energy due to the magnetic force in the pulse is

$$\frac{\mu e^2w^2}{3\delta}.$$

The energy due to the tangential electric force in the pulse is per unit volume

$$\frac{KT^2}{8\pi} = \frac{e^2w^2\sin^2\theta}{8\pi \cdot KV^2\delta^2r^2};$$

integrating this through the pulse we find that this energy is equal to $\frac{\mu e^2w^2}{3\delta}$, since $\mu K = \frac{1}{V^2}$.

Thus the total energy in the pulse is $\frac{2}{3}\frac{\mu e^2w^2}{\delta}$; and this energy radiates away into space. The energy in the field before the sphere was stopped was $\frac{1}{3}\mu e^2w^2/a$, where a is the radius of the sphere (see Art. 277). Thus if δ is not much greater than the diameter of the

sphere a very considerable fraction of the kinetic energy is radiated away when the particle is stopped.

289. Distribution of Momentum in the Field. There is no momentum inside the surface of the sphere whose radius is $b(t-T)$, there is a certain amount of momentum in the pulse, and momentum in the opposite direction in the region outside the pulse; we shall leave it as an exercise for the student to show that the momentum in the pulse is equal and opposite to that outside it, so that as soon as the sphere is reduced to rest the whole momentum in the field is zero.

290. Case of an Accelerated Charged Body. The preceding method can be applied to the case when the charged body has its velocity altered in any way, not necessarily reduced to zero. Thus if the velocity instead of being reduced to zero is diminished by δw, we can show in just the same way as before that the magnetic force H in the pulse is given by the equation

$$H = \frac{e\Delta w \cdot \sin\theta}{r\delta},$$

and the tangential electric force T by

$$T = \frac{e\Delta w \sin\theta}{KVr\delta}.$$

Now $\delta = V\delta t$ if δt is the time required to change the velocity by Δw, hence we have

$$H = \frac{e}{V}\frac{\Delta w}{\delta t}\frac{\sin\theta}{r}, \qquad T = \frac{e}{KV^2}\frac{\Delta w}{\delta t}\frac{\sin\theta}{r};$$

but $\Delta w/\delta t = -f$, where f is the acceleration of the particle, hence

$$H = -\frac{e}{V}\frac{f\sin\theta}{r}, \qquad T = -\frac{e}{KV^2}\frac{f\sin\theta}{r}.$$

It must be remembered that f is not the acceleration of the sphere at the time when H and T are estimated but at the time r/V before this. We see that when the velocity of the sphere is not uniform, part of the magnetic and electric force will vary inversely as the distance from the centre of the sphere, while the other part will vary inversely as the square of this distance; at great distances from the sphere the former part will be the most important.

The energy in the pulse emitted whilst the velocity is changing equal to

$$\frac{2}{3}\frac{e^2f^2}{V^2}d,$$

where d is the thickness of the pulse; since $d = V\delta t$, where δt is the time the acceleration lasts, the energy emitted in the time δt is

$$\frac{2}{3}\frac{e^2f^2}{V}\delta t,$$

thus the rate of emission of energy is $2e^2f^2/3V$.

291. Magnetic and Electric Forces due to a charged particle vibrating harmonically through a small distance. The magnetic force proportional to the acceleration which we have just investigated arises from the motion of the tangential part of the Faraday tubes—the portion $P'N'$ of Fig. 134; the radial tubes are however also in motion, their velocity at right angles to their length being $w\sin\theta$, where w is the velocity of the particle when its acceleration is f, i.e. at a time r/V before the force is estimated. This motion of the radial tubes produces a magnetic force $ew\sin\theta/r^2$ in the same direction as that due to the acceleration. Thus H the magnetic force at P is equal to

$$\frac{ew\sin\theta}{r^2}+\frac{ef\sin\theta}{Vr},$$

and is at right angles to OP and to the axis of z along which the particle is supposed to be moving. Let the velocity of the particle along this line be $\omega\sin pt$ and its acceleration therefore $\omega p\cos pt$. The magnetic force at P at the time t will depend upon the velocity and acceleration of the particle at the time $t-\frac{r}{V}$, these are respectively $\omega\sin p\left(t-\frac{r}{V}\right)$ and $\omega p\cos p\left(t-\frac{r}{V}\right)$, thus H the magnetic force at P is given by the equation

$$H=\frac{e\omega\sin\theta\sin p\left(t-\frac{r}{V}\right)}{r^2}+\frac{e\omega\sin\theta\, p\cos p\left(t-\frac{r}{V}\right)}{Vr}.$$

If α, β, γ are the components of this force parallel to the axes of x, y, z, then

$$\alpha=-\frac{y}{r\sin\theta}H,\qquad \beta=\frac{x}{r\sin\theta}H,\qquad \gamma=0.$$

Hence

$$\alpha = \frac{d}{dy} \frac{e\omega \sin p\left(t - \dfrac{r}{V}\right)}{r}, \quad \beta = -\frac{d}{dx} \frac{e\omega \sin p\left(t - \dfrac{r}{V}\right)}{r}.$$

If X, Y, Z are the components of the electric force, we have by equation (1), page 363,

$$K\frac{dX}{dt} = \frac{d\gamma}{dy} - \frac{d\beta}{dz} = \frac{d^2}{dx\,dz} \frac{e\omega \sin p\left(t - \dfrac{r}{V}\right)}{r},$$

$$K\frac{dY}{dt} = \frac{d\alpha}{dz} - \frac{d\gamma}{dx} = \frac{d^2}{dy\,dz} \frac{e\omega \sin p\left(t - \dfrac{r}{V}\right)}{r},$$

$$K\frac{dZ}{dt} = \frac{d\beta}{dx} - \frac{d\alpha}{dy} = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) \frac{e\omega \sin p\left(t - \dfrac{r}{V}\right)}{r}.$$

Hence the periodic parts of X, Y, Z are given by the equations

$$KX = -\frac{1}{p}\frac{d^2}{dx\,dz} \frac{e\omega \cos p\left(t - \dfrac{r}{V}\right)}{r},$$

$$KY = -\frac{1}{p}\frac{d^2}{dy\,dz} \frac{e\omega \cos p\left(t - \dfrac{r}{V}\right)}{r},$$

$$KZ = \frac{1}{p}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) \frac{e\omega \cos p\left(t - \dfrac{r}{V}\right)}{r}.$$

In addition to these there are the components

$$-\frac{e}{K}\frac{d}{dx}\frac{1}{r}, \quad -\frac{e}{K}\frac{d}{dy}\frac{1}{r}, \quad -\frac{e}{K}\frac{d}{dz}\frac{1}{r},$$

of the electrostatic force due to the charge at O. In this investigation ω is supposed to be so small compared with V that ω^2/V^2 may be neglected.

INDEX

(The numbers refer to the pages)

CAMBRIDGE : PRINTED BY J. B. PEACE, M.A., AT THE UNIVERSITY PRESS